Ph.D. Qualifying Exam (Functional Analysis)

1. (8 pts) Suppose N and F are linear subspaces of a normed linear space X, N is closed and F has finite dimension. Prove that
$$N + F = \{n + f :\ n \in N,\ f \in F\}$$
is closed.

2. (12 pts) Let H be a Hilbert space and $\{e_1, e_2, \cdots\}$ an orthonormal family in H.
   (a) State and prove the Bessel inequality.
   (b) Show that the Parseval identity is necessary and sufficient for the completeness of the orthonormal family.

3. (8 pts) Let V be a vector space and $T : V \to V$ a linear map. Suppose there is $x \in V$ such that $x,\ Tx,\ T^2x, \cdots$ span V. Prove that if $S : V \to V$ is linear and commutes with T, then there is a polynomial $p(t)$ such that $p(T) = S$.

4. (8 pts) Let $\alpha = \{\alpha_1, \alpha_2, \cdots\}$ be a sequence of complex numbers such that the series $\sum_{j=1}^{\infty} \alpha_j \beta_j$ converges for every $\{\beta_j\} \in l_q$, $1 \le q < \infty$. Prove that $\alpha \in l_p$, where $\frac{1}{p} + \frac{1}{q} = 1$.

5. (8 pts) Show that a compact operator on a Banach space maps weakly convergent sequences into norm convergent sequences.

6. (12 pts) Let M be a closed linear subspace of a normed linear space X. Show that if $M \neq X$, then there exists $f \in X \backslash M$ such that $\|f\| = 1$ and $\|f - g\| \ge \frac{1}{2}$ for all $g \in M$. Deduce that if X is infinite dimensional, then the unit sphere of X is never compact.

7. (8 pts) Let T be a self-adjoint operator on a complex Hilbert space H. Show that for any complex number α there exists λ in the spectrum of T such that
$$\|Tx - \alpha x\| \ge |\lambda - \alpha|\, \|x\| \quad \texttt{for any } x \in H.$$

8. (12 pts) Define a bounded operator $B : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ by $(Bf)(x) = \left(\tan^{-1} x\right) \cdot f(x)$. Find the spectrum of B. Is B a compact operator?

9. (16 pts)
   (a) Let X be a complex Banach space. Show that for any $x \in X$ there exists a continuous linear functional $\Lambda : X \to \mathbf{C}$ such that
$$\Lambda x = \|x\| \quad \text{and} \quad \|\Lambda\| = 1. \tag{1}$$
   (b) Show that if X is a Hilbert space there is only one linear functional for each non-zero $x \in X$ which satisfies (1).
   (c) Find a non-zero element x of the Banach space $C[0, 1]$ of complex-valued continuous functions on $[0, 1]$ with sup-norm, and give infinitely many linear functionals Λ satisfying (1).

10. (8 pts) If $x^2 = x$, $y^2 = y$ and $xy = yx$ for some x and y in a Banach algebra, prove that either $x = y$ or $\|x - y\| \ge 1$.