Please do any **FIVE** of the following problems. Each problem counts 20 points.

1. Let $f(x)=\sqrt{x^2+1}$ and the Newton's iterates $\{x_n\}, n\geq 0$, be used to approximate the solution of $f'(x)=0$. Find the basin of convergence of $\{x_n\}$.
(Hint: Find the largest set of $x_0 \in \mathbb{R}$ to ensure the convergence of $\{x_n\}$)

2. Let $\{p_n(x) \mid \deg(p_n)=n, n=0,1,\cdots\}$ be a set of orthogonal polynomials on [-1, 1] with respect to a weight function $w(x)$.
   (a) Show that p_n has exactly n distinct zeros in (-1, 1).
   (b) Suppose $w(x)=1/\sqrt{1-x^2}$. Find p_0, p_1, p_2 and p_3.

3. Suppose $f(x)\in C[0,1]$ is to be approximated by a polynomial $r(x)$. Show that the least square approximation $r_n^*(x)$ exists uniquely for any degree n>0 and $\lim_{n\to\infty}\|f(x)-r_n^*(x)\|_2=0$ on [0, 1].

4. Let $A\in\mathbb{R}^{n\times n}$ be such that $A=(1+\omega)P-(N+\omega P)$, with $P^{-1}N$ nonsingular and with eigenvalues satisfying $1\geq\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n$.
   (a) Find the values ω for which the following method
   $$(1+\omega)Px^{(k+1)}=(N+\omega P)x^{(k)}+b,\quad k\geq 0,$$
   converges to the solution of $Ax=b$, for any initial $x^{(0)}$.
   (b) What is the optimal ω for which the convergence rate in (a) is maximum.

5. Given $A\in\mathbb{R}^{n\times n}$, symmetric and positive definite. Suppose $Ax=b$ is to be solved by the following algorithm:

   Given $x_0=0, r_0=b-Ax_0=p_0, k=0$

   while $r_k\neq 0$

   $\alpha_k = r_k^T r_k / p_k^T A p_k$

   $x_{k+1}=x_k+\alpha_k p_k$

   $r_{k+1}=r_k-\alpha_k A p_k$

   End $x=x_k$

   Show that the solution is obtained within q steps if b lies in a q-dimensional invariant subspace of A.

6. Let $A \in \mathbb{R}^{n\times n}$ and symmetric.

(a) Show that A has exactly n real eigenvalues and all eigenvectors are orthogonal.

(b) If the eigenvalues satisfy $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, then verify that

$$\lambda_1 = \max_{x\in\mathbb{R}^n} \frac{x^T Ax}{x^T x} \text{ and } \lambda_n = \min_{x\in\mathbb{R}^n} \frac{x^T Ax}{x^T x}.$$

7. Given the problem $y' = f(x,y), y(0) = y_0$ where f satisfies the *Lipschitz* condition in y. Consider the numerical method

$$y_{n+1} = 4y_n - 3y_{n-1} - 2hf(x_{n-1}, y_{n-1}), \quad n \geq 1.$$

(a) Determine the order of this method.

(b) Discuss the property of convergence and stability for the given method.

8. Consider the Poisson's problem

$$-\Delta u = f(x,y), (x,y) \in R = (0,1)\times(0,1)$$

$$u = 0 \text{ on } \partial R.$$

(a) For solving the problem, derive a difference formula with uniform grid in $\overline{R}$ and determine the order of your method.

(b) Drive an iterative scheme which is able to solve the associated linear system in (a) and discuss the convergence property of your scheme with suitable assumption on $f(x, y)$.