PDE Qualifying Examination (Feb. 2006)

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Answer ALL of the following problems. Each problem carries 20 % . Here we let $\Omega \subset \mathbf{R}^n$ be a bounded, open set with a smooth boundary $\partial\Omega$

1. Consider the following problems.

   (a) Solve the first order equation $u_x + u_y + zu_z = 0$ with initial curve $x = t,\ y = 0,\ z = \sin t\ (t \in \mathbf{R})$.

   (b) Find a Fourier series solution to the equation $u_{tt} - c^2 u_{xx} = A\cos\omega t$ for $t > 0, x \in \mathbf{R}$, such that $u(0,t) = 0 = u(\pi,t)$ for $t > 0$ and $\omega \neq nc$ for all n. Hint: You may use $1 = \frac{2}{\pi}\sum_1^\infty \frac{1-(-1)^n}{n}\sin nx$ directly.

2. (a) Show that for any continuous function f, $u(x,t) = f(x - ct)$ is a weak solution in $\mathbf{R}^2$ for the equation

$$u_t + cu_x = 0 .$$

   (b) Let $u \not\equiv 0$ satisfy $u \in C^2(\mathbf{R}^n)$, $\Delta u = 0$ on $\mathbf{R}^n$. Show that $u \notin L^2(\mathbf{R}^n)$. Hint: You may use mean value property or otherwise.

3. Consider the following Cauchy problem:

$$\begin{cases} u_t = \Delta u & \text{in } \mathbf{R}^n \times (0,\infty) \\ u(x,0) = f(x) & \text{on } \mathbf{R}^n \end{cases},$$

   where f is a given continuous and bounded function on $\mathbf{R}^n$.

   (a) (5%) Write down the solution $u(x,t)$ in terms of the heat kernel $(4\pi t)^{-n/2}e^{-|x|^2/(4t)}$

   (b) (5%) Show that for each $t > 0$, the following conservation law holds

$$\int_{\mathbf{R}^n} u(x,t)\,dx = \int_{\mathbf{R}^n} f(y)\,dy .$$

(c) (10%) If $\lim_{|x|\to\infty} f(x) = 0$, show that

$$\lim_{|x|\to\infty} u(x,t) = 0 \quad \text{for fixed } t > 0 .$$

4. The Poisson integral formula for a 2-dimensional ball is

$$u(\xi) = \frac{1}{2\pi a} \int_{|x|=a} \frac{a^2 - |\xi|^2}{|x-\xi|^2} u(x)\, dS_x .$$

(a) (12%) Prove the above integral formula using Kelvin transform. You may let $R = |x - \xi|$ and $R^* = |x - \xi^*|$ for some ξ^*, and $K(x,\xi) = \frac{1}{2\pi} \ln r$ be a fundamental solution.

(b) (8%) Show that the above formula is equivalent to :

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\theta - \phi)} u(a,\phi)\, d\phi .$$

5. Let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, and

$$Lu = -\Delta u + \sum_1^n (-1)^i u_{x_i} .$$

(a) Show that $v(x) = -e^{-\lambda|x|^2} + e^{-\lambda r^2}$ for all $x \in B(0,r)$ is a strict supersolution ($Lv > 0$) when λ is large and $r/2 \le |x| \le r$.

(b) Hence or otherwise, prove the Hopf Lemma which says that, if $Lu \ge 0$ in U and $x^0 \in \partial U$ is an absolute minimum point of u in $\bar{U}$, and there exists an open ball $B \subset U$ such that $x^o \in \partial B$, then u has a strictly negative normal derivative at x^o, i.e. $\partial u/\partial\nu(x^o) < 0$.

End of Paper