Qualifying Examination in Graph Theory (2006)

1. (10 points) Prove that if G is a simple plane graph then either G has minimum degree $\delta(G) \leq 4$ or G has an edge $e = xy$ with $d(x) + d(y) \leq 11$.

2. (15 points) Prove that if an n vertex graph G has independence number 2 (i.e., the maximum independent set of G consists of two vertices), then $V(G)$ contains $m = \lceil n/3 \rceil$ disjoint subsets $V_1, V_2, \ldots, V_m$ such that each V_m induces a connected subgraph of G, and for any $1 \leq i < j \leq m$, there is an edge connecting a vertex of V_i and a vertex of V_j.

3. (15 points) Prove that for any connected graph G, for any distinct vertices u, v of G, the graph G^3 has a hamilton path connecting u and v (G^3 is the graph which has the same vertex set as G and in which xy is an edge iff G has an x-y-path of length at most 3).

4. (15 points) Prove that all the subsets of an n-element set X can be partitioned into $m = \binom{n}{\lfloor n/2 \rfloor}$ families $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$ such that for any $1 \leq i \leq m$, for any $A, B \in \mathcal{F}_i$, either $A \subset B$ or $B \subset A$.

5. (15 points) Prove that every critical k-chromatic graph (i.e., $\chi(G) = k$ and $\chi(G - v) = k - 1$ for any $v \in V(G)$) is $(k-1)$-edge connected.

6. (10 points) Suppose G is k-connected. Prove that G has a cycle of length at least $\min\{2k, V(G)\}$.

7. (10 points) A family of sets is called a Δ-system if every two of the sets have the same intersection. Prove that every infinite family of sets of cardinality at most n (for some positive integer n) contains an infinite Δ-system.

8. (10 points) Prove that a cubic graph has a nowhere zero 3-flow iff G is bipartite.