Qualify Exam: Ordinrary Differential Equations

P6 4 PA

1.(15%)

Suppose $\alpha > 0, \gamma > 0$, K, L, M are nonnegative constants and x is a nonnegative bounded continuous function satisfies

$$x(t) \leq Ke^{-\alpha t} + L\int_0^t e^{-\alpha(t-s)}x(s)ds + M\int_0^\infty e^{-\gamma s}x(t+s)ds, \quad t \geq 0.$$

If $\beta := \frac{L}{\alpha} + \frac{M}{\gamma} < 1$, show that

$$x(t) \leq (1-\beta)^{-1}Ke^{-[\alpha-(1-\beta)^{-1}L]|t|}.$$

2.(10%)

Suppose A is a real $n \times n$ matrix. Prove that the matrix equation $A^TB + BA = -C$ has a positive definite solution B for every positive definite matrix C if and only if all real part of eigenvalues of A are negative.

3.(15%)

Let E be a normed vector space, $W \subset \mathbf{R} \times E$ an open set, and $f, g : W \to E$ continuous. Suppose that for all $(t, x) \in W$, $|f(t,x) - g(t,x)| < \varepsilon$. Let K be a Lipschitz constant in x for $f(t,x)$. If $x(t), y(t)$ are solutions to $x' = f(t,x)$ and $y' = g(t,y)$, respectively, on some interval J, and $x(t_0) = y(t_0)$. Show

$$|x(t) - y(t)| \leq \frac{\varepsilon}{K}\{e^{K|t-t_0|} - 1\}.$$

4. (15%)

Consider the following system:

$$x' = x - y - x^3 \quad y' = x + y - y^3.$$

(1) Show that there is a unique equilibrium.

(2) Show that there is a unique stable limit cycle in the region

$$A = \{x \in R^2 | \ 1 \leq |x| \leq \sqrt{2}\ \}.$$