NCTU PhD Qualify Examination, Probability
September, 2009

20 points for each problem

1. Let $X_1, X_2, \cdots$, be a sequence of independent random variables. (i) prove "roughly" the Komogorove's 0–1 law, a tail event is either of probability 0 or of probability 1. (ii) prove that the event $\frac{X_1+\cdots+X_n}{n} \to 0$ has probability 0 or 1.

2. Let X be a non-negative r.v. (i) prove that $E(X^p) = p\int_{[0,\infty)} x^{p-1}P\{X > x\}dx$, for any $p > 0$. (ii) if, for some $p > 0$, $E(X^p) < \infty$, then prove that it must be $\lim_{x\to\infty} x^p P\{X > x\} = 0$.

3. One application of LLN. Assume SLLN, and let an iid sequence with a positive L^1 rv X as common distribution to represent the life-time of, say, a bulb, Let rv $N(t)$ denote the " renewal number" up to time t (you need to define it and explain your definition). State and prove the LLN for $N(t)$.

4. Let X, X_n, ξ_n are r'v.'s. (i) Prove that if $X_n \Rightarrow X$ and $\xi \to 0$ in probability, then $X_n + \xi_n \Rightarrow X$. (ii) is (i) still true if we have only $\xi_n \Rightarrow 0$?

5. Given a submartingale $(X_n, \mathcal{F}_n)$. Prove that we have one and only one decomposition $X_n = M_n + A_n$, where $(M_n, \mathcal{F}_n)$ is a martingale and A_n is an increasing process.