DEPARTMENT OF MATHEMATICS
CHIAO TUNG UNIVERSITY
Ph. D. Qualifying Examination
February, 2010
Analysis
(TOTAL 100 PTS)

1. (50%) Prove or disprove the following statements:

(a) Let $f : [0, \infty) \mapsto \mathbb{R}$ be continuous. Then f is Lebesgue integrable on $[0, \infty)$ if and only if the improper integral $\int_0^\infty f(x)dx$ exists.

(b) $L^2(X, B, \mu) \setminus L^1(X, B, \mu) \neq \phi$, where $\mu(X) = 1$.

(c) For every $\epsilon > 0$ and every Lebesgue measurable set A in $\mathbb{R}^n$, there exist an open set V and a closed set F such that $F \subset A \subset V$ and $m(V \setminus F) < \epsilon$, where m denotes the Lebesgue measure on $\mathbb{R}^n$.

(d) Let $1 < p < \infty$ and $f_n \in L^p[-\pi, \pi]$ with $\|f_n - f_{n+1}\|_p \leq 1/(n\sqrt{n})$ for all $n \geq 1$. Then f_n converges in $L^p[-\pi, \pi]$.

(e) There exists $f \in C[0, 1]$ such that $f' \in L^1[0, 1]$ and

$$\int_0^1 f'(x)dx \neq f(1) - f(0).$$

Solution:

(a) False. Consider $f(x) = \sin x/x$. Then f is not Lebesgue integrable on $[0, \infty)$, but $\displaystyle\int_0^\infty \frac{f(x)}{x}dx = \pi/2$.

(b) False. By Holder's inequality, $L^2(X, B, \mu) \subset L^1(X, B, \mu)$.

(c) True. Write $A = \cup_{m=1}^\infty A_m \cup \tilde{A}$, where $A_m = \{x \in A : m - 1 < \|x\| < m\}$ and $\tilde{A} = A \setminus \cup_m A_m$. We have $m(\tilde{A}) = 0$. Applying the regularity of m to each A_m, we can find open sets V_m and closed set F_m such that $F_m \subset A_m \subset V_m$ and $m(V_m \setminus F_m) < \epsilon/2^{m+1}$ for each m. Set $F = \cup_m F_m$ and $V = \cup_m V_m \cup \tilde{V}$, where $\tilde{V}$ is an open set containing $\tilde{A}$ with $m(\tilde{V}) < \epsilon/2$, which are the desired.

(d) True. We have $\displaystyle\sum_{n=1}^\infty \|f_n - f_{n+1}\|_p \leq \sum_{n=1}^\infty 1/(n\sqrt{n}) < \infty$

By Minkowski's inequality or the completeness of $L^p[-\pi, \pi]$, (d) follows.

(e) Consider the Canter ternary function.

2.(10%) Let ν be the Borel measure defined by

$$\nu(E) = \int_E \frac{dx}{(x^2+1)^2} \qquad \text{for all Borel subsets } E \text{ of } \mathbb{R}.$$

(a) Prove that

$$\int_{\mathbb{R}} f(x) d\nu(x) = \int_{\mathbb{R}} \frac{f(x)}{(x^2+1)^2} dx$$

for all nonnegative Borel measurable functions f on $\mathbb{R}$.

(b) Find the value of $\displaystyle\int_{\mathbb{R}} |x| d\nu(x)$.

Solution:

(a)

$$\nu(E) = \int_E \frac{dx}{(x^2+1)^2} \Longrightarrow \int_{\mathbb{R}} \chi_E d\nu(x) = \int_{\mathbb{R}} \frac{\chi_E}{(x^2+1)^2} dx$$

$$\Longrightarrow \int_{\mathbb{R}} f(x) d\nu(x) = \int_{\mathbb{R}} \frac{f(x)}{(x^2+1)^2} dx \quad \text{for all simple functions } f$$

For $f \geq 0$, choose simple functions $f_n \uparrow f$. Replace f in the last equality by f_n and then apply the Monotone Convergence Theorem. The desired result follows.

(b)

$$\int_{\mathbb{R}} |x| d\nu(x) = \int_{\mathbb{R}} \frac{|x|}{(x^2+1)^2} dx = 2\int_0^\infty \frac{x}{(x^2+1)^2} dx$$

$$= -\frac{1}{x^2+1}\bigg|_0^\infty = 1$$

3. (10%) Let $\{a_n\}_{n=0}^{\infty}$ be a complex sequence such that $\sum_{n=0}^{\infty} a_n b_n$ converges for all complex $\{b_n\}_{n=0}^{\infty} \in \ell^2$. For any positive integer N, define $T_N : \ell^2 \mapsto \mathbb{C}$ by

$$T_N(\{b_n\}) = \sum_{n=0}^{N} a_n b_n.$$

(*a*) Prove that $T_N \in (\ell^2)^*$ and $\|T_N\| = (\sum_{n=0}^{N} |a_n|^2)^{1/2}$.

(*b*) Is $\{a_n\}_{n=0}^{\infty} \in \ell^2$? Why?

Solution: (a)

$$\left|T_N(\{b_n\})\right| \le \sum_{n=0}^{N} |a_n b_n| \le \left(\sum_{n=0}^{N} |a_n|^2\right)^{1/2} \left(\sum_{n=0}^{N} |b_n|^2\right)^{1/2}$$
$$\Longrightarrow T_N \in (\ell^2)^*$$

Moreover, $\|T_N\| \le (\sum_{n=0}^{N} |a_n|^2)^{1/2}$. Consider $b_n = \bar{a_n}$. We get the equality.

Remark: We can get (*a*) by using Riesz representation Theorem.

(b) follows from the uniform boundedness principle by considering the family $\{T_N\}_{N=1}^{\infty}$, where $T_N : \ell^2 \mapsto \mathbb{C}$.

4. (10%) Set $f_n(t) = e^{int}$, where $n \ge 1$.

(*a*) Prove that $f_n \to 0$ weakly in $L^2[-\pi, \pi]$.

(*b*) Is $f_n \to 0$ in L^2-norm? Why?

Solution: (a) For any $g \in L^2[-\pi, \pi]$, choose step functions $g_m = \chi_{[c,d]}$ such that $\|g_m - g\|_2 \longrightarrow 0$. We have

$$\int_{-\pi}^{\pi} f_n(x) g_m(x) dx = \int_{c}^{d} e^{inx} dx \longrightarrow 0 \quad \text{as} \quad n \to \infty$$

and

$$\left|\int_{-\pi}^{\pi} f_n(x) g(x) dx - \int_{-\pi}^{\pi} f_n(x) g_m(x) dx\right| \le \int_{-\pi}^{\pi} |g(x) - g_m(x)| dx$$
$$\le (2\pi)^{1/2} \|g - g_m\|_2 \longrightarrow 0$$

(b)

$$\|f_n - 0\|_2^2 = \int_{-\pi}^{\pi} |e^{inx}|^2 dx = 2\pi \nrightarrow 0$$

5. (10%) Let $T \in (C[0,1])^*$ such that $T(1+x+\cdots+x^n)=0$ for all $n \geq 0$. Prove that $T=0$.

Solution: For any $p(x)=a_0+a_1x+\cdots+a_nx^n$, we have

$$p(x)=\sum_{k=0}^{n} a_k((1+x+\cdots+x^k)-(1+x+\cdots+x^{k-1})).$$

By the linearity of T and the hypothesis,

$$T(p(x))=\sum_{k=0}^{n} a_k(T(1+x+\cdots+x^k)-T(1+x+\cdots+x^{k-1}))=0.$$

We know that the set of polynomials is dense in $C[0,1]$, so by the continuity of T, $T=0$.

6. (10%) Define the function Tf by the formula:

$$Tf(y)=\int_{-\infty}^{\infty}\frac{f(x)\sqrt{x^2+1}}{x^2+y^2+1}\,dx \qquad (y\in\mathbb{R}).$$

(a) Prove that if $f\in L^1(\mathbb{R})$, then $Tf\in L^1(\mathbb{R})$.

(b) Find the value $\sup\limits_{\|f\|_1\neq 0}\dfrac{\|Tf\|_1}{\|f\|_1}$.

Solution: By Fubini's theorem, we get

$$\begin{aligned}
\|Tf\|_1 &= \int_{-\infty}^{\infty}|Tf(y)|dy \leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|f(x)|\sqrt{x^2+1}}{x^2+y^2+1}\,dxdy\\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\frac{1}{x^2+y^2+1}\,dy\right)|f(x)|\sqrt{x^2+1}\,dx\\
&= \int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{x^2+1}}\tan^{-1}\Big(\frac{y}{\sqrt{x^2+1}}\Big)\Big|_{-\infty}^{\infty}\right)|f(x)|\sqrt{x^2+1}\,dx=\pi\|f\|_1
\end{aligned}$$

So (a) follows.

(b) The above argument (considering $f\geq 0$) also implies

$$\sup_{\|f\|_1\neq 0}\frac{\|Tf\|_1}{\|f\|_1}=\pi$$