Differential Equations Qualifying Exam, Feb. 2010

This exam contains 9 problems with a total of 100 points. The first 4 problems are related to ODE and the second 5 problems are related to PDE. Give your arguments as clear as possible.

1. (20 points) Given $f : \mathbb{R} \to \mathbb{R}$ continuous so that

$$f(0) = 0, \qquad f(x) \neq 0 \text{ for } x \neq 0. \tag{1}$$

Consider the IVP (initial value problem)

$$\begin{cases} \dfrac{dx}{dt}(t) = f(x(t)) \\ x(0) = 0. \end{cases} \tag{2}$$

Note that the equation is **separable** and $x(t) \equiv 0$ is a solution to the initial value problem. Do the following problems.

(a). (5 points) Show that if $x(t)$ is a solution to the IVP on the interval $[0, a]$ and $x(a) = 0$, then $x(t) = 0$ for all $0 \le t \le a$.

(b). (10 points) Show that the IVP has a **unique** solution (i.e., the only solution is $x(t) \equiv 0$) if and only if the improper integrals

$$\int_{-\varepsilon}^{0} \frac{1}{f(x)} dx \quad \text{and} \quad \int_{0}^{\varepsilon} \frac{1}{f(x)} dx \tag{3}$$

both diverge for some $\varepsilon > 0$.

(c). (5 points) Assume that $f(x) > 0$ on $(0, \infty)$ and $\int_0^\infty \frac{1}{f(x)} dx$ exists. Show that the IVP has a nonzero solution $x(t)$ whose maximal time interval of existence on the part $t > 0$ is a **finite** interval $[0, T)$ with $\lim_{t \to T} x(t) = \infty$, where T is given by $T = \int_0^\infty \frac{1}{f(x)} dx$. Moreover, we have $x(t) > 0$ on $(0, T)$.

Solution:

(a). Assume $x(t)$ is not identically zero on $[0, a]$. Then there exists some $t_0 \in (0, a)$ such that $x(t_0) > 0$ (we may assume so) and some interval $(\alpha, \beta) \subset (0, a)$ with $x(t) > 0$ on (α, β), $x(\alpha) = x(\beta) = 0$. Now we have

$$\beta - \alpha = \int_\alpha^\beta 1 dt = \int_\alpha^\beta \frac{x'(t)}{f(x(t))} dt = \int_{x(\alpha)}^{x(\beta)} \frac{1}{f(x)} dx = 0$$

which is a contradiction. Hence $x(t)$ must be identically zero on $[0, a]$.

(b). ($\Longrightarrow$) Assume the solution is unique (hence the only solution is $x(t) \equiv 0$) and $\int_0^\varepsilon \frac{1}{f(x)} dx$ converges. We shall derive a contradiction (if the other one $\int_{-\varepsilon}^0 \frac{1}{f(x)} dx$ converges, the proof is similar). Define the function

$$F(x) = \int_0^x \frac{1}{f(s)} ds, \quad F'(x) = \frac{1}{f(x)} \neq 0, \quad x \in (0, \varepsilon).$$

Since the improper integral converges, $F(x)$ is continuous on $[0, \varepsilon]$ with $F(0) = 0$. Without loss of generality we may assume that $F'(x) > 0$ on $(0, \varepsilon)$ with $F'(0+) = +\infty$. For each $t \in [0, F(\varepsilon)]$, there is a unique $x(t) \in [0,\varepsilon]$ such that $F(x(t)) = t$. By the **inverse function theorem**, such $x(t)$ is differentiable on $t \in (0,\varepsilon)$ with

$$x'(t) = \frac{1}{F'(x(t))} = f(x(t)), \quad x(0) = 0, \quad x(t) > 0 \text{ on } (0,\varepsilon).$$

This gives a contradiction.

($\Longleftarrow$) On the other hand, assume that both integrals in (3) diverge but the solution is not unique. We shall derive a contradiction. Now we may assume there exists some solution $x(t)$ to the IVP with $x(0) = 0,\ x(t) > 0$ on $(0,a]$ for some $a > 0$. Hence

$$a = \int_0^a \frac{x'(t)}{f(x(t))} dt = \int_0^{x(a)} \frac{1}{f(x)} dx \quad \text{(this is divergent)}$$

and we get a contradiction.

(c). Let $F(x) = \int_0^x \frac{1}{f(s)} ds,\ x \in (0,\infty)$. Then $F(x)$ is strictly increasing on $[0,\infty)$ with $F(0) = 0,\ F(\infty) = T := \int_0^\infty \frac{1}{f(x)} dx$ (this is a finite positive number). By the **inverse function theorem**, we have

$$\int_0^{x(t)} \frac{1}{f(s)} ds = t \quad \text{for all} \quad t \in (0,T).$$

Such $x(t)$ is differentiable on $(0,T)$ with $x'(t) = \frac{1}{f(x(t))} > 0,\ x(0) = 0,\ \lim_{t\to T} x(t) = \infty$, and $x(t) > 0$ on $(0,T)$. □

2. (10 points) Prove the following *general version* of **Gronwall inequality** (it will contain most of the other familiar Gronwall inequalities as special cases). This general version is very useful in ODE theory. Let $f(t),\ g(t),\ \psi(t)$ be three continuous functions on $[a,b]$ with $\psi(t) \geq 0$ on $[a,b]$. If

$$f(t) \leq g(t) + \int_a^t \psi(s) f(s) ds \quad \text{for all} \quad t \in [a,b] \tag{4}$$

then show that

$$f(t) \leq g(t) + \int_a^t \left[g(s)\psi(s)\exp\left(\int_s^t \psi(u) du\right)\right] ds \quad \text{for all} \quad t \in [a,b]. \tag{5}$$

In particular, when $g(t) = C$ is a constant, show that the above implies the familiar inequality

$$f(t) \leq C\exp\left(\int_a^t \psi(u) du\right) \quad \text{for all} \quad t \in [a,b]. \tag{6}$$

<u>**Solution**</u>:

By approximation if necessary, we may assume that $\psi(t) > 0$ on $[a,b]$. Let

$$\Phi(t) = \int_a^t \psi(s) f(s) ds, \qquad \Phi(a) = 0.$$

Then $\Phi'(t) = \psi(t) f(t)$ and

$$\frac{\Phi'(t)}{\psi(t)} = f(t) \leq g(t) + \Phi(t), \quad t \in [a,b]. \tag{7}$$

Multiply (7) by $\psi(t)\exp\left(-\int_a^t \psi(u)\,du\right)$ to get

$$\Phi'(t)\exp\left(-\int_a^t \psi(u)\,du\right) \le [g(t)+\Phi(t)]\,\psi(t)\exp\left(-\int_a^t \psi(u)\,du\right)$$

which implies

$$\frac{d}{dt}\left[\Phi(t)\exp\left(-\int_a^t \psi(u)\,du\right)\right] \le g(t)\,\psi(t)\exp\left(-\int_a^t \psi(u)\,du\right), \quad t\in[a,b]. \tag{8}$$

Integrate both sides of (8) on $[a,t]$ to get (note that $\Phi(a)=0$)

$$\Phi(t)\exp\left(-\int_a^t \psi(u)\,du\right) \le \int_a^t \left(g(s)\,\psi(s)\exp\left(-\int_a^s \psi(u)\,du\right)\right) ds$$

and so

$$\Phi(t) \le \left[\exp\left(\int_a^t \psi(u)\,du\right)\right]\cdot\left[\int_a^t \left[g(s)\,\psi(s)\exp\left(-\int_a^s \psi(u)\,du\right)\right] ds\right].$$

The above can be written as

$$\begin{aligned}\Phi(t) &\le \int_a^t \left[g(s)\,\psi(s)\exp\left(\int_a^t \psi(u)\,du - \int_a^s \psi(u)\,du\right)\right] ds\\ &= \int_a^t \left[g(s)\,\psi(s)\exp\left(\int_s^t \psi(u)\,du\right)\right] ds.\end{aligned}$$

By the assumption

$$f(t) \le g(t)+\Phi(t) \le g(t) + \int_a^t \left[g(s)\,\psi(s)\exp\left(\int_s^t \psi(u)\,du\right)\right] ds$$

the proof of (5) is done.

Next, when $g(t)=C$ is a constant, (5) becomes

$$\begin{aligned}f(t) &\le C + C\int_a^t \left[\psi(s)\exp\left(\int_s^t \psi(u)\,du\right)\right] ds = C - C\int_a^t \frac{d}{ds}\left[\exp\left(\int_s^t \psi(u)\,du\right)\right] ds\\ &= C - C\left[1-\exp\left(\int_a^t \psi(u)\,du\right)\right] = C\exp\left(\int_a^t \psi(u)\,du\right).\end{aligned}$$

The proof is done. □

3. (10 points) Let $f(\mathbf{x})$ be a smooth vector field on $\mathbb{R}^n$ such that it generates a smooth dynamical system $\varphi_t(\mathbf{x})$ on $\mathbb{R}^n$ (i.e., the map $\mathbb{R}^n \times (-\infty,\infty) \to \mathbb{R}^n$ given by $(t,\mathbf{x}) \to \varphi_t(\mathbf{x})$ is smooth). Let $D \subset \mathbb{R}^n$ be a measurable set and let

$$D_t = \{\varphi_t(\mathbf{x}) : \mathbf{x}\in D\} \subset \mathbb{R}^n.$$

Show that for any time $t>0$, the volume of D_t is given by

$$vol(D_t) = \int_D \exp\left(\int_0^t (div\ f)(\varphi_s(\mathbf{x}))\,ds\right) d\mathbf{x} \tag{9}$$

where $div\ f$ denotes the divergence of the vector field f. PS: In doing this problem, you can take the well-known "**Liouville theorem**" for granted.

__Solution__:

By change of variables formula for integration, we have

$$vol\left(D_t\right)=\int_{D_t} 1d\mathbf{x}=\int_{D_0}\left|\det J\varphi_t\left(\mathbf{x}\right)\right|d\mathbf{x} \tag{10}$$

where $J\varphi_t(\mathbf{x})$ is the Jacobi matrix of the flow $\varphi_t : D_0 \to D_t$ (this is a diffeomorphism). For fixed $\mathbf{x} \in D_0$ denote the Jacobi matrix $J\varphi_t(\mathbf{x})$ by $J(t)$. By **variation formula** it satisfies the equation

$$\begin{cases} \dfrac{dJ}{dt}(t) = (Df)(\varphi_t(\mathbf{x}))\, J(t) \\ J(0) = Id \quad (n \times n \text{ identity matrix}) \end{cases}$$

and by **Liouville theorem** we have

$$\det J(t) = \exp\left(\int_0^t Trace\left[(Df)(\varphi_s(\mathbf{x}))\right] ds\right)$$

where

$$Trace\left[(Df)(\varphi_s(\mathbf{x}))\right] = (div\ f)(\varphi_s(\mathbf{x})).$$

Hence (10) becomes

$$\begin{aligned} vol\left(D_t\right) &= \int_{D_0}\left|\det J\varphi_t(\mathbf{x})\right| d\mathbf{x} = \int_{D_0} \exp\left(\int_0^t Trace\left[(Df)(\varphi_s(\mathbf{x}))\right] ds\right) d\mathbf{x} \\ &= \int_D \exp\left(\int_0^t (div\ f)(\varphi_s(\mathbf{x}))\, ds\right) d\mathbf{x}. \end{aligned}$$

The proof is done. □

4. (15 points)

(a) (5 points) Assume that A is a real $n \times n$ constant matrix. Find a necessary and sufficient condition on A so that for any two solutions $X^{(1)}(t),\ X^{(2)}(t) \in \mathbb{R}^n$ to the first order linear system

$$\frac{dX}{dt} = AX, \quad X = X(t) \in \mathbb{R}^n, \quad t \in [0, \infty) \tag{11}$$

their inner product $\left\langle X^{(1)}(t),\ X^{(2)}(t)\right\rangle$ is independent of time $t \in [0, \infty)$.

(b) (10 points) Consider the second order scalar ODE

$$x''(t) + x(t) = g(t), \quad t \in (-\infty, \infty), \tag{12}$$

where $g(t)$ is a given smooth 2π**-periodic function** on $(-\infty, \infty)$. In general, a solution to (12) may not be 2π-periodic even when g is 2π-periodic (for example $g(t) = \cos t$ and $x(t) = \frac{1}{2} t \sin t$). Show that if $g(t)$ satisfies the condition

$$\int_0^{2\pi} g(t)\cos t dt = \int_0^{2\pi} g(t) \sin t dt = 0 \tag{13}$$

then any solution $x(t)$ to (12) on $(-\infty, \infty)$ is also 2π-periodic.

Solution:

(a). The necessary and sufficient condition on A is $A+A^T=0$. To see this, compute

$$\frac{d}{dt}\left\langle X^{(1)}(t),\ X^{(2)}(t)\right\rangle = \left\langle AX^{(1)}(t),\ X^{(2)}(t)\right\rangle + \left\langle X^{(1)}(t),\ AX^{(2)}(t)\right\rangle = \left\langle \left(A+A^T\right)X^{(1)}(t),\ X^{(2)}(t)\right\rangle. \tag{14}$$

It is easy to see that (14) is zero for any two solutions $X^{(1)}(t),\ X^{(2)}(t)$ if and only if $A+A^T=0$.

(b). Fix $t_0\in(-\infty,\infty)$ and assume $x(t_0)=x_0,\ x'(t_0)=x_0'$. From ODE theory (using the variation of parameter method), the solution satisfying the initial condition is given by the nice formula

$$x(t)=x_0\cos(t-t_0)+x_0'\sin(t-t_0)+\int_{t_0}^{t}g(s)\sin(t-s)\,ds,\quad t\in(-\infty,\infty). \tag{15}$$

We have

$$\begin{aligned}
&x(t+2\pi)-x(t)\\
&=-\cos t\cdot\int_t^{t+2\pi}g(s)\sin s ds+\sin t\cdot\int_t^{t+2\pi}g(s)\cos s ds\\
&=-\cos t\cdot\int_0^{2\pi}g(s)\sin s ds+\sin t\cdot\int_0^{2\pi}g(s)\cos s ds
\end{aligned}$$

and so if the periodic function $g(s)$ satisfies condition (13), then $x(t)$ is also 2π-periodic on $(-\infty,\infty)$.□

5. (5 points) Consider the first order quasilinear equation in two variables (x,y) :

$$a(x,y,u)\,u_x+b(x,y,u)\,u_y=c(x,y,u),\quad u=u(x,y) \tag{16}$$

where the given functions $a(x,y,z),\ b(x,y,z)$ and $c(x,y,z)$ are assumed to be smooth everywhere and assume that we have two smooth integral surfaces (i.e., solution surfaces) represented by the following two smooth solutions

$$S_1: z=u(x,y),\qquad S_2: z=v(x,y),$$

which intersect transversally along a smooth curve $\gamma(t)$ given by

$$x(t)=e^t,\quad y(t)=\ln(t+5),\quad z(t)=\sin t,\quad t\in[0,\infty).$$

If we know that $a(\gamma(t))=a(x(t),y(t),z(t))=t^2+1$, find $b(\gamma(t))$ and $c(\gamma(t))$.

Solution:

By the theory of first order quasilinear equation, we know that $\gamma'(t)$ is **parallel** to the vector field $(a(\gamma(t)),\ b(\gamma(t)),\ c(\gamma(t)))$ for all $t\in[0,\infty)$. Hence there exists a function $\lambda(t)$ such that

$$\begin{cases}
\dfrac{dx}{dt}=\lambda(t)\,a(x(t),y(t),z(t))\\
\dfrac{dy}{dt}=\lambda(t)\,b(x(t),y(t),z(t))\\
\dfrac{dz}{dt}=\lambda(t)\,c(x(t),y(t),z(t))
\end{cases}$$

for all $t\in[0,\infty)$. By the information given, we know that $\lambda(t)=e^t/\left(t^2+1\right)$. Hence

$$b(\gamma(t))=\frac{t^2+1}{e^t}\frac{1}{t+5}\qquad\text{and}\qquad c(\gamma(t))=\frac{t^2+1}{e^t}\cos t,\quad t\in[0,\infty).$$

□

6. (10 points) The **Laplace operator** $\triangle$ has the effect of **averaging** (there are essentially infinitely many ways to see this). To realize this, you are required to do the following problem. Assume $u(x_1, ..., x_n) : \mathbb{R}^n \to \mathbb{R}$ is a smooth function and let

$$f_\sigma(t) := u(t\sigma), \quad t \in (-\infty, \infty), \tag{17}$$

where σ is an unit vector in $\mathbb{R}^n$, i.e., $|\sigma| = 1$, $\sigma \in S^{n-1}$. Show that we have the **average formula**

$$\frac{1}{|S^{n-1}|} \int_{S^{n-1}} f''_\sigma(0)\, d\sigma \text{ (this is surface integral on } S^{n-1}) = \frac{1}{n} (\triangle u)(0), \tag{18}$$

where $|S^{n-1}|$ is the surface area of S^{n-1}.

Solution:We have

$$f'_\sigma(0) = Du(0) \cdot \sigma, \qquad f'_\sigma(t) = Du(t\sigma) \cdot \sigma$$
$$f''_\sigma(0) = \left.\frac{\partial^2 u}{\partial x_i \partial x_j}\right|_{x=0} \sigma_i \sigma_j \text{ (sum over } i,\ j), \qquad \sigma = (\sigma_1, ..., \sigma_n) \in S^{n-1} \subset \mathbb{R}^n$$

and so

$$\frac{1}{n\omega_n} \int_{S^{n-1}} f''_\sigma(0)\, d\sigma = \frac{1}{n\omega_n} \int_{S^{n-1}} \langle A\sigma, \sigma \rangle\, d\sigma \tag{19}$$

where $n\omega_n$ is the surface measure of S^{n-1} and $A : \mathbb{R}^n \to \mathbb{R}^n$ is the linear map given by $Ax = Mx,\ x \in \mathbb{R}^n$, where

$$M = \left[\left.\frac{\partial^2 u}{\partial x_i \partial x_j}\right|_{x=0} \right] = \text{the symmetric Hessian matrix of } u \text{ at } 0. \tag{20}$$

Viewing Ax as a vector field on $\mathbb{R}^n$, we have

$$div\ (Ax) = trace\ A = \triangle u(0).$$

Applying the divergence theorem to (19), we get

$$\frac{1}{n\omega_n} \int_{S^{n-1}} \langle A\sigma, \sigma \rangle\, d\sigma = \frac{1}{n\omega_n} \int_B div\ (Ax)\, dx = \frac{1}{n\omega_n} (\triangle u(0))\, \omega_n. \tag{21}$$

where ω_n is the volume of the unit ball $B = B_1(0)$ in $\mathbb{R}^n$. From (21), we conclude the average formula

$$\frac{1}{n\omega_n} \int_{S^{n-1}} f''_\sigma(0)\, d\sigma = \frac{1}{n} (\triangle u)(0), \qquad f_\sigma(t) = u(t\sigma). \tag{22}$$

□

7. (10 points)

(a) (5 points) Let $U \in \mathbb{R}^n$ be a bounded domain with smooth boundary and let $U_T = U \times (0, T] \in \mathbb{R}^{n+1}$, $\Gamma_T = \overline{U_T} - U_T$ (Γ_T is the space-time parabolic boundary of U_T). Consider the parabolic initial/boundary value problem

$$\begin{cases} \partial_t u - \Delta u = f & \text{in} \quad U_T \\ u = g & \text{on} \quad \Gamma_T \end{cases} \tag{23}$$

where $f(x,t)$ and $g(x,t)$ are given smooth functions on $\overline{U_T}$ and $\overline{\Gamma_T}$ respectively. Use "**energy method**" to show that (23) has at most one solution u in the space $C^2\left(\overline{U_T}\right)$.

(b) (5 points) Consider the nonlinear parabolic PDE

$$\begin{cases} \partial_t u - \Delta u = h(u) & \text{in} \quad U_T \\ u = g & \text{on} \quad \Gamma_T \end{cases} \tag{24}$$

where $h(z) : \mathbb{R} \to \mathbb{R}$ is a given smooth function and we assume that the boundary function $g(x,t) = g(x)$ is independent of time. Assume that $u \in C^2\left(\overline{U_T}\right)$ is a solution to (24). Show that the functional

$$E(t) := \frac{1}{2} \int_U |\nabla u(x,t)|^2 \, dx - \int_U H(u(x,t)) \, dx, \quad H'(z) = h(z) \tag{25}$$

is decreasing in $t \in [0, \infty)$. We call it a **Lyapunov functional** for the parabolic equation (24).

Solution:

(a). It suffices to such that the only solution $w \in C^2\left(\overline{U_T}\right)$ to the following

$$\begin{cases} \partial_t w - \Delta w = 0 & \text{in} \quad U_T \\ w = 0 & \text{on} \quad \Gamma_T \end{cases}$$

is the zero solution $w \equiv 0$. For each $t \in (0, T]$, we compute

$$\frac{d}{dt} \int_U w^2(x,t) \, dx = \int_U 2w(x,t) \frac{\partial w}{\partial t}(x,t) \, dx = \int_U 2w(x,t) \Delta w(x,t) \, dx = -2 \int_U |\nabla w(x,t)|^2 \, dx \leq 0,$$

where in the last equality we have used the divergence theorem and the fact that $w = 0$ on Γ_T. Hence

$$\int_U w^2(x,t) \, dx \leq \int_U w^2(x,0) \, dx = 0 \quad \text{for all} \quad t \in (0,T]$$

and so $w(x,t) \equiv 0$ for all $(x,t) \in \overline{U_T}$.

(b). We first compute

$$E'(t) = \int_U \nabla \left(\frac{\partial u}{\partial t} \right) \cdot \nabla u dx - \int_U h(u) \left(\frac{\partial u}{\partial t} \right) dx.$$

By the divergence theorem we have

$$0 = \int_U div \left(\frac{\partial u}{\partial t} \nabla u \right) dx = \int_U \nabla \left(\frac{\partial u}{\partial t} \right) \cdot \nabla u dx + \int_U \left(\frac{\partial u}{\partial t} \right) \Delta u dx,$$

where the first equality is due to the identity

$$\int_U div \left(\frac{\partial u}{\partial t} \nabla u \right) dx = \int_{\partial U} \frac{\partial u}{\partial t} \left(\nabla u \cdot \mathbf{n}_{out} \right) d\sigma, \quad t \in (0,T]$$

and the fact

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial g(x)}{\partial t} = 0, \quad x \in \partial U.$$

Now we have

$$E'(t) = -\int_U \left(\frac{\partial u}{\partial t} \right) \Delta u dx - \int_U h(u) \left(\frac{\partial u}{\partial t} \right) dx = -\int_U \left(\frac{\partial u}{\partial t} \right)^2 dx \leq 0.$$

The proof is done. □

8. (10 points) Let L be a second order linear operator of the form

$$Lu=\sum_{i,\ j=1}^{n}a_{ij}\left(x,t\right)\frac{\partial^{2}u}{\partial x_{i}\partial x_{j}}+\sum_{i=1}^{n}b_{i}\left(x,t\right)\frac{\partial u}{\partial x_{i}}+c\left(x,t\right)u-\frac{\partial u}{\partial t},\qquad (x,t)\in U_{T}=U\times(0,T],$$

where U is a bounded domain in $\mathbb{R}^n$ and all coefficients are continuous on $\bar{U}_T$. We assume that the equation is uniformly parabolic in the sense that there exists a constant $\theta>0$ such that

$$\sum_{i,\ j=1}^{n}a_{ij}\left(x,t\right)\xi_{i}\xi_{j}\geq\theta\left|\xi\right|^{2},\quad\forall\,\xi\in\mathbb{R}^{n},\quad\forall\ (x,t)\in U_{T}.$$

The classical maximum principle says that if $u\in C_1^2\left(U_T\right)\cap C^0\left(\bar{U}_T\right)$ satisfies $Lu=0$ in U_T, and $c\left(x,t\right)$ satisfies the sign condition $c\leq 0$ in U_T, then

$$\max_{\bar{U}_T}|u|\leq\max_{\Gamma_T}|u|\,,\tag{26}$$

where $\Gamma_T=\overline{U_T}-U_T$ is the space-time parabolic boundary of U_T. Show that if we have replace the condition by $c\leq\varepsilon$ in U_T ($\varepsilon>0$ is a constant), then (26) becomes

$$\max_{\bar{U}_T}|u|\leq e^{\varepsilon T}\cdot\max_{\Gamma_T}|u|\,.\tag{27}$$

Hint: consider $v\left(x,t\right)=f\left(t\right)u\left(x,t\right)$ for some suitable function $f\left(t\right)$.

Solution:

Let $v\left(x,t\right)=e^{-\varepsilon t}u\left(x,t\right)$, then v satisfies

$$\tilde{L}v=0,\qquad\tilde{L}=L-\varepsilon.$$

The $\tilde{c}$ for $\tilde{L}$ satisfies $\tilde{c}\left(x,t\right)\leq 0$ in U_T. Hence

$$e^{-\varepsilon T}\max_{\bar{U}_T}|u|\leq\max_{\bar{U}_T}|v|=\max_{\Gamma_T}|v|\leq\max_{\Gamma_T}|u|\,.$$

□

9. (10 points) The Maxwell equations are given by

$$\begin{cases}\mathbf{E}_t=curl\ \mathbf{B}\\ \mathbf{B}_t=-curl\ \mathbf{E}\\ div\ \mathbf{B}=div\ \mathbf{E}=0,\end{cases}\tag{28}$$

where $\mathbf{E}\left(x_1,x_2,x_3,t\right):\mathbb{R}^3\times(-\infty,\infty)\to\mathbb{R}^3$ and the same for $\mathbf{B}\left(x_1,x_2,x_3,t\right)$. Show that if $\mathbf{E}=(E^1,E^2,E^3)$ and $\mathbf{B}=(B^1,B^2,B^3)$ solve the Maxwell equations, then u satisfies the wave equation

$$u_{tt}\left(x,t\right)-\triangle u\left(x,t\right)=0,\quad x=(x_1,x_2,x_3)\tag{29}$$

where $u=E^i$ or B^i $\ (i=1,2,3)$.

Solution:

For convenience, just look at $u=E^1$. We have

$$E_t^1=\frac{\partial B^3}{\partial x_2}-\frac{\partial B^2}{\partial x_3}$$

and then

$$\begin{aligned}
E^1_{tt} &= \frac{\partial}{\partial x_2} B^3_t - \frac{\partial}{\partial x_3} B^2_t \\
&= -\frac{\partial}{\partial x_2}\left(\frac{\partial E^2}{\partial x_1} - \frac{\partial E^1}{\partial x_2}\right) - \frac{\partial}{\partial x_3}\left(\frac{\partial E^3}{\partial x_1} - \frac{\partial E^1}{\partial x_3}\right) \\
&= \frac{\partial^2 E^1}{\partial x_2^2} + \frac{\partial^2 E^1}{\partial x_3^2} - \frac{\partial}{\partial x_2}\frac{\partial E^2}{\partial x_1} - \frac{\partial}{\partial x_3}\frac{\partial E^3}{\partial x_1} \\
&= \frac{\partial^2 E^1}{\partial x_2^2} + \frac{\partial^2 E^1}{\partial x_3^2} - \frac{\partial}{\partial x_1}\left(\frac{\partial E^2}{\partial x_2} + \frac{\partial E^3}{\partial x_3}\right) \quad (\text{note that } \mathit{div}\ \mathbf{E} = 0) \\
&= \frac{\partial^2 E^1}{\partial x_2^2} + \frac{\partial^2 E^1}{\partial x_3^2} - \frac{\partial}{\partial x_1}\left(-\frac{\partial E^1}{\partial x_1}\right) = \triangle E^1.
\end{aligned}$$

The proof is done. □