# 交通大學應用數學系博士班資格考(2010 年 9 月)

## PhD Qualifying Exam in Numerical Analysis
## Fall 2010

1. (20%) Let $\varphi$ be a real-valued continuous function defined on $[a,b]$. Assume that $\varphi([a,b]) \subseteq [a,b]$, $\varphi'$ exists on $(a,b)$ and $\exists\ 0 < k < 1$ such that $|\varphi'(x)| \le k$ for all $x \in (a,b)$. Consider the fixed point iterations given by $x_{n+1} = \varphi(x_n)$ for $n \ge 0$.

   (a) Show that $\{x_n\}$ converges to the unique fixed point $p$ of $\varphi$ for any $x_0 \in [a,b]$.

   (b) Assume that $\varphi^{(r)}$ is continuous and $\varphi^{(k)}(p) = 0$ for $1 \le k < r$ but $\varphi^{(r)}(p) \ne 0$. Show that the fixed point iteration method converges with order $r$.

   (c) Let $\varphi(x) := x - \dfrac{f(x)}{f'(x)}$ for some smooth function $f$. Assume that $f(p) = 0$ and $f'(p) \ne 0$. Show that under suitable assumptions, the order of convergence of Newton's method for solving $f(x) = 0$ is two.

2. (20%) Let $f$ be a real-valued function defined on $[a,b]$. Assume that $f \in C^2[a,b]$.

   (a) Please use the Lagrange interpolation to derive the trapezoidal formula with an error term for $\int_a^b f(x)dx$.

   (b) Let $a = x_0 < x_1 < \cdots < x_n = b$ be a uniform partition of $[a,b]$ with mesh size $h = (b-a)/n$. Prove that for such uniform partition the error term for the composite trapezoidal formula is

$$\int_a^b f(x)dx - \frac{h}{2}\left(f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right) = -\frac{1}{12}(b-a)h^2 f''(\xi), \quad \text{for some } \xi \in (a,b).$$

3. (15%) Let $f$ be sufficiently smooth and satisfy the Lipschitz condition such that there exists a unique solution $x(t)$ for $t_0 \le t \le t_0 + T$ of the following initial value problem:

$$\text{(IVP)} \quad \begin{cases} x'(t) &= f(t, x(t)) \quad \text{for } t_0 < t < t_0 + T, \\ x(t_0) &= x_0 \in \mathbb{R}. \end{cases}$$

   (a) Derive the second-order Taylor-series method for the numerical approximation of the IVP.

   (b) Derive the Heun method which is a second-order Runge-Kutta method for the numerical approximation of the IVP in the following form:

$$x(t+h) = x(t) + \frac{h}{2}f(t,x) + \frac{h}{2}f(t+h, x + hf(t,x)) + O(h^3).$$

4. (15%) Consider the linear system $Ax = b$, where $A \in \mathbb{R}^{n\times n}$ is a given nonsingular matrix and $b \in \mathbb{R}^n$ is a given vector.

   (a) Describe the basic concept of linear iterative method by using the so-called preconditioning matrix (or splitting matrix) $P$ such that $A = P - N$, where $P$ and $N$ are two suitable matrices and $P$ is nonsingular.

   (b) What are the preconditioning matrices for the Jacobi method and the Gauss-Seidel method?

   (c) Prove that if $\|I - P^{-1}A\| < 1$ for some subordinate matrix norm, then the sequence generated by the linear iterative method in part (a) converges to the solution of $Ax = b$ for any initial vector $x^{(0)}$.

5. (15%) Consider the following two-point boundary value problem:

$$\text{(BVP)} \quad \begin{cases} -\varepsilon u''(x) + \beta u'(x) = f(x) & \text{for} \quad 0 < x < 1, \\ u(0) = 0 \ \text{ and } \ u(1) = 0, \end{cases}$$

where $\varepsilon$ and $\beta$ are two positive constants and $f$ is a given smooth function. Let $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ be a uniform partition of $[0, 1]$ with mesh size $h > 0$.

(a) Derive the second-order central difference method for the BVP.

(b) Derive the finite element method using piecewise linear elements for the BVP.

(c) Please use the "mean-value theorem for integrals" to explain that the above-mentioned methods are essentially the same method for the BVP.

6. (15%) Consider the following initial-boundary value problem of the 1-D heat equation:

$$\text{(IBVP)} \quad \begin{cases} u_t - u_{xx} &= 0, \quad t > 0,\ 0 < x < 1, \\ u(x,0) &= u_0(x), \quad 0 \le x \le 1, \\ u(0,t) &= u(1,t) \ = \ 0, \quad t > 0. \end{cases}$$

(a) Find an explicit finite difference method to solve the IBVP and discuss the stability properties of the method.

(b) Construct an implicit finite difference method to improve the stability of the explicit method developed in part (a).

Hint: the eigenvalues of the $(n-1) \times (n-1)$ tridiagonal matrix $2I - U - U^{\top}$ are given by $\mu_i = 4\sin^2\left(\frac{i\pi}{2n}\right)$ for $i = 1, 2, \cdots, n-1$, where $I$ is the $(n-1) \times (n-1)$ identity matrix and $U$ is given by

$$U = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & & 1 \\ & & & & 0 \end{bmatrix}_{(n-1)\times(n-1)}$$

and all unspecified entries are 0.