交通大學應用數學系博士班資格考(2010 年 9 月)

Probability, SEP 2010

20 points for each problem

1)Let $\{X_n\}$ be a sequence of identically distributed random variables with finite expectation. Prove that

$$\lim_{n\to\infty}\frac{1}{n}\mathrm{E}\{\max_{1\le j\le n}|X_j|\}=0.$$

(Hint: For $X\ge 0$, $\mathrm{E}X=\int_0^\infty P\{X\ge x\}dx$.)

2)Let $\{X_n\}$ be *i.i.d.* random variables with $\mathrm{E}|X_1|<\infty$ and $S_n=\sum_{i=1}^n X_i$. By the strong law of large numbers S_n/n converges *a.s.* to $\mathrm{E}X_1$. Prove that in fact S_n/n is uniformly integrable and converges to $\mathrm{E}X_1$ in L^1.

3)Let $X(t)$, $t\ge 0$, be such that for any bounded stopping time τ, $X(\tau)$ is integrable and $\mathrm{E}X(\tau)=\mathrm{E}X(0)$. Prove that $X(t)$, $t\ge 0$ is a martingale.

4)Interpret and prove probabilistically the trigonometric identity

$$\frac{\sin t}{t}=\prod_1^\infty \cos\frac{t}{2^n}.$$

(Hint: Use characteristic function.)

5)Let $X_0,X_1,\dots,X_t,\dots$ be a Markov chain generated by an irreducible transition matrix $P=\{p_{i,j}\}$ with the stationary (invariant) distribution π in a finite state space $\{1,2,\cdots,N\}$. Define iteratively

$$\tau_0=\min_{s\ge 0}\{s|X_s\ne 1\} \quad\text{and}\quad \tau_{t+1}=\min_{s>\tau_t}\{s|X_s\ne 1\}.$$

Define $\{Y_t\}$ as

$$Y_t=X_{\tau_t}.$$

Show that $\{Y_t\}$ is a Markov chain on the state space $\{2,3,\cdots,N\}$ with the transition matrix $Q=\{q_{i,j}\}$,

$$q_{i,j}=p_{i,j}+\frac{p_{i,1}\cdot p_{1,j}}{1-p_{1,1}},\qquad 2\le i\le N,$$

and the stationary (invariant) distribution μ,

$$\mu_i=\frac{\pi_i}{1-\pi_1},\qquad 2\le i\le N.$$