博士班資格考

代數

Qualifying Exam

FEBRUARY 24, 2011

Instructions. You need to show all your work in order to get full credit. When using a theorem, you must state it clearly and correctly.

1. *(10 points)* Let p be an odd prime and $R = \mathbb{Z}/2p\mathbb{Z}$. Consider $f(x) = x^p - x$. What are the roots of f in R?

2. *(10 points)* Let A and B be $n \times n$ matrix over $\mathbb{R}$. Suppose that B is invertible and $A^tBA = B$. Prove that λ is an eigenvalue of A if and only if $\frac{1}{\lambda}$ is an eigenvalue of A.

3. *(10 points)* Let A and B be two $n \times n$ complex matrices. Suppose that A and B have the same characteristic polynomials. Do A and B have the same eigenvalues? If yes, prove it. If not, find a counterexample.

4. *(15 points)* Let ξ be a primitive 8-th root of unity and consider the field extension $\mathbb{Q} \subseteq \mathbb{Q}(\xi)$.

(i) Find the Galois group of this extension.

(ii) How many intermediate fields are there between $\mathbb{Q}$ and $\mathbb{Q}(\xi)$.

5. *(10 points)* Prove that $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{R}) = \{e\}$.

6. *(10 points)* Let R be a commutative Noetherian ring with identity. Let x be an indeterminate. Prove that the ring of polynomials $R[x]$ is Noetherian.

7. *(15 points)* Let G be a group of order 12 such that G contains no element of order 6.

(i) Prove that its 2-sylow subgroup is a normal subgroup of G.

(ii) Prove that its 2-sylow subgroup is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

8. *(20 points)* Let $\mathbb{F}_q$ denote the finite field of $q = p^m$ elements, where p is a prime. Let $\mathrm{GL}_n(\mathbb{F}_q)$ denote the group of invertible $n \times n$ matrices with coefficients in $\mathbb{F}_q$.

(i) What is the order of $\mathrm{GL}_n(\mathbb{F}_q)$?

(ii) What is the order of a p-sylow subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$? Find such a p-sylow subgroup.

(iii) A *flag* in $\mathbb{F}_q^n$ sequence of vector spaces V_i of dimension i of the form

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq \mathbb{F}_q^n.$$

What is the cardinality of the set of flags?