DEPARTMENT OF MATHEMATICS
CHIAO TUNG UNIVERSITY
Ph. D. Qualifying Examination
Feb, 2011
Analysis
(TOTAL 100 PTS)

Throughout this exam, $|E|$ denotes the Lebesgue measure of E.

1. (50%) Prove or disprove the following statements:

(a) Let $1 < p < 2$ and $f \in L^p(\mathbb{R})$. Then $f = g + h$ for some $g \in L^1(\mathbb{R})$ and $h \in L^2(\mathbb{R})$.

(b) Let $f : [0,1] \mapsto \mathbb{R}$ be absolutely continuous. If $|f'(x)| \leq 1$ a.e. on $(0,1]$, then $\left|\frac{f(x)-f(0)}{x-0}\right| \leq 1$ for all $0 < x < 1$.

(c) Let $\{E_n\}$ be a sequence of Lebesgue measurable sets with the property that $\sum_{n=1}^{\infty} |E_n| < \infty$. Then $|\limsup E_n| = 0$, where

$$\limsup E_n = \cap_{m=1}^{\infty}\left(\cup_{n=m}^{\infty} E_n\right).$$

(d) Let $f \in L^1(\mathbb{R})$ and $a > 0$. Then

$$\int_{-\infty}^{\infty}\left|\frac{1}{2a}\int_{x-a}^{x+a} f(t)dt\right|dx \leq \|f\|_1.$$

(e) Let ν be the Borel measure defined by

$$\nu(E) = \int_E x^2 dx \qquad \text{for all Borel sets } E.$$

Then given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|\nu(E)| < \epsilon \quad \text{whenever} \quad |E| < \delta.$$

2.(10%) Let $\phi \in L^1(\mathbb{R}^n)$. Set $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon),\ \epsilon > 0$. Prove that

$$\lim_{\epsilon \to 0}\int_{\{\|x\|>M\}} \phi_\epsilon(x)dx = 0 \qquad \text{for all } M > 0.$$

3.(10%) Let $f_n, f \in L^2[-\pi,\pi]$. Suppose that $f_n \longrightarrow f$ in $L^2[-\pi,\pi]$. Prove that

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f_n(x)f_{n+1}(x)dx = \int_{-\pi}^{\pi}(f(x))^2dx.$$

4.(10%) Consider the operator $Tf(x)$ defined, at least formally, by

$$Tf(x) = \int_{-\infty}^{\infty} \frac{f(y)}{x^2 + y^2 + 1} dy, \qquad x \in \mathbb{R}.$$

Does $f \in L^2(\mathbb{R})$ imply that $Tf \in L^1(\mathbb{R})$? If so, find the value

$$\sup_{\|f\|_2 \neq 0} \frac{\|Tf\|_1}{\|f\|_2}.$$

5. (10%) Let c denote the set of all convergent sequences. Is it possible to find some $T \in (\ell^\infty)^*$ with the following property:

$$T(\{a_n\}) = \lim_{n \to \infty} \frac{a_1 + a_2 + \cdots + a_n}{n} \quad \text{for all } \{a_n\} \in c?$$

Give your reason.

6. (10%) Let $T \in C[0,1]^*$ with the property:

$$T(x^n) = \int_0^1 \frac{x^n}{\sqrt{1+x^2}} dx \qquad (n = 1, 2, \cdots).$$

Prove that $T(f) = T(f(0)) + \int_0^1 \frac{f(x) - f(0)}{\sqrt{1+x^2}} dx$ for all $f \in C[0,1]$.