Department of Applied Mathematics
National Chiao Tung University
Ph.D. Qualifying Examination
Discrete Mathematics
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Problem 1.(15%) Let G be a bipartite graph with vertices $X \cup Y$ such that every edge of G has one endpoint in X and one in Y. Use Dilworth's theorem to prove that if $|\Gamma(A)| \geq |A|$ for every $A \subseteq X$, the G has a complete matching from X to Y in G.

Problem 2.(15%)
Show that a $k \times n$ Latin rectangle, $k < n$, can be extended to a $(k+1) \times n$ Latin rectangle.

Problem 3.(15%) State and prove Brooks' theorem.

Problem 4.(15%) Consider walks in the xy-plane where each step is either R : $(x, y) \to (x+1, y)$ or $U : (x, y) \to (x, y+1)$. If we start at $(0, 0)$ how many ways can we reach $(2n, 2n)$ without passing through one of the points $(2i-1, 2i-1), i = 1, 2, \ldots, n$. Prove that this number is the Catalan number u_{2n+1}.

Problem 5.(15%) Let M be an 11×11 matrix with all entries $0, 1$ with the following properties: (i) every row of M has six ones; (ii) the inner product of any two distinct rows of M is at most 3. Show that M is the incidence matrix of a 2-$(11, 6, 3)$ design, and further M is unique (up to isomorphism).

Problem 6.(15%) Find a subset $S = \{s_1, s_2, \ldots, s_5\}$ of $\mathbb{Z}_{21}$ such that the elements of $\mathbb{Z}_{21}$ as points and the 21 blocks $S + x$ $(x \in \mathbb{Z}_{21})$ form a projective plane of order 4. Show your work (don't just write out the S).

Problem 7.(10%) Show that if the edges of K_{17} are colored with three colors, there must be a monochromatic triangle.