博士班資格考

Probability, FEB 2011

20 points for each problem

1)(i)Let X_s be Poisson with parameter $s > 0$. Compute the characteristic function of X_s and show that for $s \to \infty$,

$$\frac{X_s - s}{\sqrt{s}} \text{ converges weakly (in distribution) to a standard normal distribution.}$$

(ii)X and Y are *i.i.d.* random variables with mean zero and variance one. Suppose that $\frac{X+Y}{\sqrt{2}}$ and X have the same distribution, show that X and Y must be standard normal.(Hint: use the central limit theorem.)

2)Find the following conditional expectations and verify your answers.

(i)X and Y are exchangeable random variables, i.e. (X, Y) and (Y, X) are identically distributed, what is $E(X \mid X+Y)$?

(ii)X and Y are jointly normal random variables with

$$EX = EY = 0, \quad EX^2 = EY^2 = 1, \quad EXY = \rho, \quad \text{what is} \quad E(X \mid Y)?$$

3)Let $\{X_n\}$ be identically distributed with $E \mid X_1 \mid < \infty$. Show that for $n \to \infty$,

$$\frac{max_{k \leq n} \mid X_k \mid}{n} \to 0 \text{ almost surely (a.s.). (Hint:} \frac{\mid X_n \mid}{n} \to 0 \text{ a.s.)}$$

Furthermore if $\{X_n\}$ are *i.i.d.* and $EX_1 \neq 0$, show that

$$\frac{\sum_{k \leq n} X_k^2}{(\sum_{k \leq n} X_k)^2} \to 0 \text{ a.s.}$$

4)Show that a supermartingale $M(t), 0 \leq t \leq T$, is a martingale if $EM(T) = EM(0)$.

5)Let $X_0, X_1, \cdots, X_n, \cdots$ be a stochastic process with a countable state space E. Show that the Markovian property of the process is equivalent to the following property: for any

$$r \geq 1, \quad l_1 < l_2 \cdots < l_r < m < n, \quad \text{and} \quad i_1, \cdots, i_r, j, k \in E, \quad \text{we have}$$

$$P\{X_{l_1} = i_1, \cdots, X_{l_r} = i_r, X_n = k \mid X_m = j\}$$
$$= P\{X_{l_1} = i_1, \cdots, X_{l_r} = i_r \mid X_m = j\} P\{X_n = k \mid X_m = j\}.$$