NATIONAL CHIAO TUNG UNIVERSITY

2019 Real Analysis Spring Ph.D. Qualifying Exam

1. Let $f, f_k : [0,1] \longrightarrow \mathbb{R}$ be Lebesgue measurable functions such that $\lim_{k\to\infty} \int_0^1 |f_k(x) - f(x)|\, dx = 0$ and that $\sup_{k\in\mathbb{N}} \int_0^1 |f_k(x)|^4\, dx < \infty$.
   (a) (5 %) Show that $\int_0^1 |f(x)|^4\, dx < \infty$.
   (b) i. (5 %) Show that $\int_0^1 |f_k(x)|^2\, dx < \infty$ for all $k \in \mathbb{N}$ and that $\int_0^1 |f(x)|^2\, dx < \infty$.
   ii. (5 %) Furthermore, show that $\lim_{k\to\infty} \int_0^1 |f_k(x) - f(x)|^2\, dx = 0$

2. (11 %) Prove or disprove that if $f_k : [0,1] \longrightarrow \mathbb{R}$ are continuous functions for all $k \in \mathbb{N}$ and if $\lim_{k\to\infty} f_k(x) = 0$ for all $x \in [0,1]$, then $\lim_{k\to 0} \int_0^1 f_k(x)\, dx = 0$.

3. Write $|A|$ as the Lebesgue measure of a Lebesgue measurable set $A \subset \mathbb{R}$. Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of Lebesgue integrable functions defined on $\mathbb{R}$ and suppose that $\lim_{k\to\infty} \int_{\mathbb{R}} |f_k(x)|\, dx = 0$. For any $\lambda > 0$, let $E_k(\lambda) = \{x \in \mathbb{R} \mid |f_k(x)| \geq \lambda\}$. **Prove or disprove** the following statements.
   (a) (6 %) There is a Lebesgue measurable subset $E \subset \mathbb{R}$ such that $|E| = 0$, and that $\lim_{k\to\infty} f_k(x) = 0$, for every $x \notin E$.
   (b) (6 %) For any $\lambda > 0$, $\lim_{k\to\infty} \sqrt{\lambda}|E_k(\lambda)| = 0$.
   (c) (6 %) $\lim_{k\to\infty} \dfrac{|E_k\left(\frac{1}{k}\right)|}{k} = 0$.
   (d) (6 %) $\lim_{k\to\infty} \dfrac{|E_k\left(\frac{1}{k}\right)|}{\sqrt{k}} = 0$.

4. Write $|A|_e$ as the outer measure of a set $A \subset \mathbb{R}^n$. **Prove or disprove** the following statements.
   (a) (10 %) Let E_k be a monotonic sequence of sets in $\mathbb{R}^n$ with each $|E_k|_e < +\infty$, then
   $$|\lim_{k\to\infty} E_k|_e = \lim_{k\to\infty} |E_k|_e.$$
   (b) (10 %) Two sets $A, B \subset \mathbb{R}^n$ have measurable union $A \cup B$ and $|A \cup B| = |A|_e + |B|_e$. Then A and B are both measurable.

5. (15 %) Let $f : \mathbb{R} \cup \{\pm\infty\} \longrightarrow \mathbb{R}$ be an increasing function on the real line $\mathbb{R}$ with $f(-\infty) = 0$ and $f(\infty) = 1$. Prove that f is absolutely continuous on every closed finite interval **if and only if**
$$\int_{\mathbb{R}} f'(x)\, dx = 1$$

6. (15 %) Suppose $f \in L^1(0,1)$. Find

$$\lim_{k\to\infty} \int_0^1 k \ln\left(1 + \frac{|f(x)|^2}{k^2}\right) dx$$

Hint: You may need to prove the elementary inequality $\ln(1+t) \leq at^b$ for some $a, b > 0$ and for $t > 0$.