NATIONAL CHIAO TUNG UNIVERSITY

Real Analysis Ph.D. Qualifying Exam, Fall 2019

1. Let μ be a measure on X and $\{E_k\}_{k\in\mathbb{N}}$ be a sequence of μ-measurable subset of X. Prove or disprove (by giving a counterexample) the following statements:
   (a) (10 %) If $\sum_{k=1}^{\infty}\mu(E_k)<\infty$, then for μ-almost all $x\in X$ belongs to *at most finitely many* E_k's.
   (b) (10 %) If $\mu(\{x\in X \mid x\in E_k \text{ for infinitely many } k\in\mathbb{N}\})=0$, then $\sum_{k=1}^{\infty}\mu(E_k)<\infty$.

2. (15 %) Let $f:\mathbb{R}\longrightarrow\mathbb{R}$. Suppose that both $f(x)$ and $xf(x)$ are integrable with respect to the Lebesgue measure dx. Define
$$g(y)=\int_{\mathbb{R}} f(x)\sin(xy)\,dx.$$
Prove that g is differentiable and
$$g'(y)=\int_{\mathbb{R}} xf(x)\cos(xy)\,dx.$$

3. (15 %) Let $f:[0,1]\longrightarrow\mathbb{R}$ so that $f\in L^1([0,1])$ with respect to the Lebesgue measure. Which of the following three functions: $\sqrt{|f|}$, f^2, or $\arctan f\in L^1([0,1])$? Justify your answers.

4. (20 %) Show that $L^p(\mathbb{R}^n)(1\le p\le\infty)$is a complete space.

5. (15 %) Show that the unit ball $S=\{x\in X|\ ||x||\le 1\}$ of a Banach space X is compact if and only if X is of finite dimensional.

6. (15 %) A Banach space X is said to be uniformly convex if for any $\varepsilon>0$ there exists a $\delta>0$ such that for $f,g\in X$, if $||f||=1$, $||g||=1$ and $||f+g||>2-\delta$ implies that $||f-g||<\varepsilon$. Show that any Hilbert space is uniformly convex.