On latin cubes with prescribed intersections

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1. Introduction

A latin cube C of order v is a v-tuple \((L_1, L_2, \ldots, L_v)\) of pairwise disjoint latin squares of order \(v\). Let \(C = (L_1, L_2, \ldots, L_v)\) and \(D = (M_1, M_2, \ldots, M_v)\) be two latin cubes of order \(v\) (with the same entries), then the intersection of \(C\) and \(D\) is defined to be the number

\[|C \cap D| = \sum_{i=1}^{v} |L_i \cap M_i|,\]

where \(|L_i \cap M_i|\) is the number of common entries of \(L_i\) and \(M_i\). Moreover, we define \(J[v]\) as the set of positive integers \(k\) such that there exist two latin cubes of order \(v\) with intersection \(k\), and we define \(I[v] = \{0, 1, 2, \ldots, v^3-14\} \cup \{v^3-12, v^3-8, v^3\}\).

In [3] results on \(J[v]\) were used in solving the intersection problem for Steiner quadruple systems of order \(4v\), where \(v\) is the order of a Steiner quadruple system, and \(v \geq 10\). Some of the results concerning \(J[v]\) which were obtained in that paper are the following:

1. \(J[10] \supseteq I[10] \setminus \{10^3-21, 10^3-14\}\).
2. \(J[v] \supseteq I[v] \setminus \{v^3-21, v^3-14\}\) for every even \(v \geq 20\).

In this paper we prove that \(J[v] = I[v]\) for every \(v \geq 24\) and \(J[v] \supseteq I[v] \setminus \{v^3-14\}\) when \(20 \leq v \leq 23\).

2. Main theorems

It is easy to show that the intersection of two latin squares of order \(v\) cannot be \(v^2-5, v^2-3, v^2-2,\) and \(v^2-1\). Hence we have the following lemma.

**Lemma 2.1.** \(J[v] \subseteq I[v]\) for every order \(v\).

**Proof.** It is well known that a latin cube is equivalent to a 3-quasigroup \(Q\) [2], and the set \(\{(x, y, z) \mid (x, y, z) \in Q\}\), with one component fixed, corresponds to a latin square. Since the intersections of two latin squares cannot be \(v^2-5, v^2-3, v^2-2,\) and \(v^2-1\), we conclude that the intersections of two latin cubes of order \(v\) cannot be \(v^3-13, v^3-11, v^3-10, v^3-9, v^3-7, \ldots, v^3-1\). This implies that \(J[v] \subseteq I[v]\).

**Lemma 2.2.** \(v^3-21 \in J[v]\) for every \(v \geq 6\).
Proof. It is well known [1] that the partial latin square $A$ of order 3 (Figure 2.1) can be embedded in a latin square $L = \ell_{i,j}$ of order $v \geq 6$. Let $M = [m_{i,j}]$ be a latin square of order $v$ containing the subsquare $B$ (Figure 2.1) in the upper-left corner. We construct a latin cube $C = (L_1, L_2, ..., L_v)$ by letting $L_1 = L_2 = \ell_{i,j}^{t}$, $t = 2, 3, ..., v$, $\ell_{i,j}^{t} = (\ell_{i,j})^{t}$, and $\alpha_{t} = \left(m_{1,1}^{t} m_{1,2}^{t} \cdots m_{1,v}^{t}\right)$. It is easy to see that $C$ is a latin cube which contains the partial latin cube $D$ (Figure 2.2) in the upper-left corner of $L_1, L_2, L_3$. We can replace $D$ by $D'$ (Figure 2.2), and denote the new latin cube as $C'$. The theorem then follows as $|C \cap C'| = v^3 - 21$.

![Figure 2.1](image1.png)

![Figure 2.2](image2.png)

Lemma 2.3. $v^3 - 14 \in J[v], v \geq 24$. 

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Proof. Since the partial latin cube $E$ (Figure 2.3) can be embedded in a latin cube $L$ of order 12 (Figure 2.4), and a latin cube of order $n$ can be embedded in a latin cube of order $m \geq 2n$ [3], then the partial latin cube $E$ can be embedded in a latin cube of any order $v \geq 24$. We can replace $E$ by $E'$ (Figure 2.3), this concludes the proof.

$$E = \begin{array}{ccc}
1 & 2 & \\
2 & 1 & 3 \\
3 & 2 & \\
\end{array} \hspace{1cm} E' = \begin{array}{ccc}
2 & 1 & \\
1 & 3 & 2 \\
2 & 3 & \\
\end{array}$$

Figure 2.3

$$A_1 \begin{array}{cccccc}
1 & 2 & 4 & 3 & 5 & 6 \\
2 & 1 & 3 & 5 & 6 & 4 \\
4 & 3 & 2 & 6 & 1 & 5 \\
3 & 5 & 6 & 1 & 4 & 2 \\
5 & 6 & 1 & 4 & 2 & 3 \\
6 & 4 & 5 & 2 & 3 & 1 \\
\end{array} \hspace{1cm} A_4 \begin{array}{cccccc}
5 & 6 & 3 & 1 & 4 & 2 \\
3 & 5 & 6 & 4 & 2 & 1 \\
1 & 4 & 5 & 2 & 3 & 6 \\
4 & 2 & 1 & 5 & 6 & 3 \\
6 & 1 & 2 & 3 & 5 & 4 \\
2 & 3 & 4 & 6 & 1 & 5 \\
\end{array}$$

$$A_2 \begin{array}{cccccc}
2 & 1 & 5 & 4 & 6 & 3 \\
1 & 3 & 2 & 6 & 4 & 5 \\
6 & 2 & 3 & 1 & 5 & 4 \\
5 & 6 & 4 & 2 & 3 & 1 \\
3 & 4 & 6 & 5 & 1 & 2 \\
4 & 5 & 1 & 3 & 2 & 6 \\
\end{array} \hspace{1cm} A_5 \begin{array}{cccccc}
12 & 10 & 2 & 5 & 9 & 1 \\
6 & 4 & 7 & 2 & 5 & 9 \\
3 & 5 & 1 & 4 & 6 & 2 \\
7 & 3 & 5 & 12 & 2 & 4 \\
1 & 8 & 4 & 6 & 3 & 5 \\
5 & 2 & 6 & 1 & 4 & 3 \\
\end{array}$$
$$L = (L_1, L_2, \ldots, L_{12}), \ B_k(i, j) = \begin{cases} A_k(i, j) + n, & \text{if } A_k(i, j) \leq n, \\ A_k(i, j) - n, & \text{if } A_k(i, j) > n. \end{cases}$$

$$L_k = \begin{array}{cc}
A_k & B_k \\
B_k & A_k
\end{array}, \quad k = 1, 2, 3, 4.$$

$$L_5 = \begin{array}{cc}
A_5 & B_5 \\
A_6 & B_6
\end{array}$$

$$L_6 = \begin{array}{cc}
A_6 & B_6 \\
A_5 & B_5
\end{array}$$

$$L_7 = \begin{array}{cc}
B_5 & A_6 \\
B_8 & A_8
\end{array}$$

$$L_8 = \begin{array}{cc}
B_5 & A_6 \\
B_5 & A_5
\end{array}$$

$$L_{k+8} = \begin{array}{cc}
B_k & A_k \\
A_k & B_k
\end{array}, \quad k = 1, 2, 3, 4.$$
Lemma 2.4. \( J[10] \supseteq I[10] \setminus \{10^3-14\} \).

Proof. By Lemma 2.2 and the results obtained in [3].

For convenience of the following lemma, we denote the set \( \{a+b \mid a \in A \text{ and } b \in B\} \) by \( A + B \).

Lemma 2.5. \( J[v] \supseteq I[v] \setminus \{v^3-14\} \) for every \( v \), \( 20 \leq v \leq 39 \).

Proof. Since a latin cube of order \( n \) can be embedded in a latin cube of order \( m \geq 2n \) [3], let \( C \) be a latin cube of order \( v \), \( 20 \leq v \leq 39 \), containing a subcube \( B \) of order 10. \( B \) can, of course, be removed and replaced by any other latin cube on the same symbols. Now the following three parts of \( C \) can be permuted independently:

1. the entries 1,2,...,10 in the right-lower corner of \( L_1,L_2,...,L_{10} \),
2. the entries 1,2,...,10 but not in \( B \) or (1),
3. the entries 11,12,...,\( v \).

By applying the permutation to (1), (2), and (3) independently, we have
\[
J[v] \supseteq J[10] + \{0,10(v-10),20(v-10),...,80(v-10),100(v-10)\} + \{0,(v-10)v,2(v-10)v,...,8(v-10)v,10(v-10)v\} + \{0,v^2,2v^2,...,(v-12)v^2,(v-10)v^2\}.
\]
Since \( 20 \leq v \leq 39 \), it follows by Lemma 2.4 that \( J[v] \supseteq I[v] \setminus \{v^3-14\} \).

Lemma 2.6. If \( J[v] \supseteq I[v] \setminus \{v^3-14\} \), then \( J[2v] \supseteq I[2v] \setminus \{(2v)^3-14\} \), and \( J[2v+1] \supseteq I[2v+1] \setminus \{(2v+1)^3-14\} \), for every \( v \geq 10 \).

Proof. It is similar to Lemma 2.5.

Lemma 2.7. \( J[v] \supseteq I[v] \setminus \{v^3-14\} \) for every \( v \geq 20 \).

Proof. By Lemma 2.5, and 2.6.

Now we have the following theorem.

Theorem 2.8. \( J[v] = I[v] \) for every \( v \geq 24 \).

Proof. It is a direct result of Lemmas 2.3 and 2.7.
References


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