MULTICOLORED PARALLELISMS OF ISOMORPHIC SPANNING TREES

S. AKBARI†, A. ALIPOUR†, H. L. FU‡, AND Y. H. LO‡

Abstract. A subgraph in an edge-colored graph is multicolored if all its edges receive distinct colors. In this paper, we prove that a complete graph on $2^m$ ($m \neq 2$) vertices $K_{2^m}$ can be properly edge-colored with $2^m - 1$ colors in such a way that the edges of $K_{2^m}$ can be partitioned into $m$ multicolored isomorphic spanning trees.

Key words. complete graph, multicolored tree, parallelism

AMS subject classifications. 05B15, 05C05, 05C15, 05C70

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A spanning subgraph of a graph $G$ is a subgraph $H$ with $V(H) = V(G)$. A proper $k$-edge coloring of a graph $G$ is a mapping from $E(G)$ into a set of colors $\{1, \ldots, k\}$ such that incident edges of $G$ receive distinct colors. An $h$-total-coloring of a graph $G$ is a mapping from $V(G) \cup E(G)$ into a set of colors $\{1, \ldots, h\}$ such that (i) adjacent vertices in $G$ receive distinct colors, (ii) incident edges in $G$ receive distinct colors, and (iii) any vertex and its incident edges receive distinct colors. The edge chromatic number of a graph $G$ is the minimum number $k$ for which $G$ has a proper $k$-edge coloring. Throughout this paper $K_m$ and $K_{m,n}$ denote the complete graph of order $m$ and the complete bipartite graph with partite sets of sizes $m$ and $n$, respectively. It is well known that the edge chromatic number of $K_m$ is $m$ if $m$ is odd, and $m - 1$ if $m$ is even [7, p. 15]. Assume that $m$ is a natural number. For any integer $i$ we denote the residue of $i$ modulo $m$ in the set $\{1, \ldots, m\}$ by $[i]_m$. The following result is known.

Lemma 1 (see [7, p. 16]). If $m$ is an odd positive integer, then $K_m$ has an $m$-total coloring.

A Latin square of order $m$ is an $m \times m$ array of $m$ symbols in which every symbol occurs exactly once in each row and column of the array. A Room square of side $2m - 1$ is a $(2m - 1) \times (2m - 1)$ array whose cells are empty or contain an unordered pair of distinct integers chosen from $R = \{1, \ldots, 2m\}$, such that the entries of a given row contain every member of $R$ precisely once, and similarly for columns, and the array contains every unordered pair of members of $R$ precisely once. Room squares have been found for all odd $2m - 1 \geq 7$ [2, p. 239]. An example of a Room square of side 7 is shown in Table 1.

A subgraph in an edge-colored graph is said to be multicolored if no two edges have the same color. Using a Room square of side $2m - 1$ one may obtain a proper
edge coloring of $K_{2m}$ with $2m - 1$ colors in which all edges can be partitioned into $2m - 1$ multicolored perfect matchings. For example, using the rows of Table 1 we give a proper edge coloring of $K_8$ with 7 colors. We denote the vertices of $K_8$ by $1, \ldots, 8$. In Table 1, if $rs$ appears in the $i$th row, then we color the edge $rs$ with color $i$. For instance, the edges 47, 16, 38, 25 are colored with color 4. Each column in Table 1 corresponds to a multicolored perfect matching of $K_8$. In a recent paper [1] the existence of the multicolored matchings in an arbitrary edge-colored complete graph has been studied. A Latin square of order $m$ corresponds to a proper edge coloring of $K_{m,m}$ with $m$ colors. Indeed if $L = (L_{ij})$ is a Latin square of order $m$ and $\{u_1, \ldots, u_m\}$ and $\{v_1, \ldots, v_m\}$ are two parts of $K_{m,m}$, then we color the edge $u_iv_j$ with $L_{ij}$. Since $L$ has $m$ symbols, we have an $m$-edge coloring of $K_{m,m}$, and since every symbol occurs exactly once in each row and each column of $L$, the edge coloring is proper. Also the existence of two orthogonal Latin squares of order $m$ corresponds to a proper edge coloring of $K_{m,m}$ with $m$ colors for which all edges can be partitioned into $m$ multicolored perfect matchings. For example, suppose that $L = (L_{ij})$ and $R = (R_{ij})$ are two orthogonal Latin squares of order $m$ with symbols of the set $\{1, \ldots, m\}$, and $\{u_1, \ldots, u_m\}$ and $\{v_1, \ldots, v_m\}$ are two parts of $K_{m,m}$. As we saw before, the function $c$, where $c(u_iv_j) = L_{ij}$, is a proper $m$-edge coloring of $K_{m,m}$. For any $r$, $1 \leq r \leq m$, let $M_r$ be the set of all edges $u_iv_j$ such that $R_{ij} = r$. Obviously $\{M_1, \ldots, M_m\}$ is an edge partition of $E(K_{m,m})$. Since the symbol $r$ occurs exactly once in each row and each column of $R$, $M_r$ is a perfect matching, and since $L$ and $R$ are orthogonal, if $R_{ij} = r$, then the symbols $L_{ij}$ are distinct and we conclude that $M_r$ is multicolored. There is a classic result which says that for any natural number $m$, $m \neq 2, 6$, there exist two orthogonal Latin squares of order $m$; see [3].

We say that the complete graph $K_{2m}$ admits a multicolored tree parallelism (MTP) if there exists a proper edge coloring of $K_{2m}$ with $2m - 1$ colors for which all edges can be partitioned into $m$ isomorphic multicolored spanning trees. It is clear that the complete graph $K_4$ does not admit an MTP. We note here that such a partition of the edges of $K_{2m}$ can be viewed as a parallelism as defined in [5] by Cameron, with an additional property due to edge colors. In fact, finding a partition as obtained above corresponds to an arrangement of the edges of $K_{2m}$ into an array of $2m - 1$ rows and $m$ columns such that each row contains the edges with the same color which form a perfect matching and the edges in each column form a multicolored spanning tree of $K_{2m}$; moreover, all the $m$ spanning trees are isomorphic. Therefore, the partition creates a double parallelism of $K_{2m}$, one from the rows of the perfect matchings and the other from the columns of the edge disjoint isomorphic spanning trees. The following result has been proven in [6].

**Theorem A** (see [6]). If $m \neq 1, 3$ and $K_{2m}$ admits an MTP, then for any $r \geq 1$, $K_{2r,m}$ admits an MTP.

There exist three interesting conjectures on the edge partitioning of the complete graphs into multicolored spanning trees.

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Fig. 1.

Constantine’s Conjecture (weak version; see [6]). For any natural number $m$, $m > 2$, $K_{2m}$ admits an MTP.

Brualdi–Hollingsworth Conjecture (see [4]). If $m > 2$, then in any proper edge coloring of $K_{2m}$ with $2m - 1$ colors, all edges can be partitioned into $m$ multicolored spanning trees.

In [4] it was proved that in any proper edge coloring of $K_{2m}$ ($m > 2$) with $2m - 1$ colors there are at least two edge disjoint multicolored spanning trees.

Constantine’s Conjecture (strong version; see [6]). If $m > 2$, then in any proper edge coloring of $K_{2m}$ with $2m - 1$ colors, all edges can be partitioned into $m$ isomorphic multicolored spanning trees.

The main goal of this paper is to prove the first conjecture.

Example 1. The complete graph $K_6$ admits an MTP. To see this consider the complete graph $K_6$ with the vertex set $\{1, \ldots, 6\}$. Table 2 gives a proper edge coloring of $K_6$ with colors $c_1, \ldots, c_5$ as well as an MTP for it. The $i$th row of this table is the set of all edges with color $c_i$. Each column denotes the edges of a multicolored spanning tree. Figure 1 shows that the spanning trees $T_1, T_2, T_3$ are isomorphic.

In [6] it has been shown that $K_8$ admits an MTP.

Using the software Gap, Peter Cameron found a decomposition of $K_{6,6}$ into six isomorphic multicolored graphs $K_{1,3} \cup 3K_2 \cup 2K_1$. In the next lemma, using Cameron’s decomposition we find an MTP for $K_{12}$.

Lemma 2. The complete graph $K_{12}$ admits an MTP.

Proof. Consider the complete graph $K_{12}$ with the vertex set $\{a_1, \ldots, a_m, v_1, \ldots, v_6\}$. Table 3 gives a proper edge coloring of $K_{12}$ with colors $c_1, \ldots, c_{11}$ as well as an MTP for it. The $i$th row of this table is the set of all edges with color $c_i$. Each column denotes the edges of a multicolored spanning tree. Note that the first six rows of the table determine a decomposition of $K_{6,6}$ into six multicolored subgraphs isomorphic to $K_{1,3} \cup 3K_2 \cup 2K_1$. \( \square \)

Now, we are ready to prove our main result.

Theorem. For $m \neq 2$, $K_{2m}$ admits an MTP.

Proof. First suppose that $m$ is an odd integer. Consider the complete graph $K_{2m}$ defined on the set $A \cup B$ where $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_m\}$. For
convenience, let $G$ and $H$ be the complete graphs on the sets $A$ and $B$, respectively. Since $m$ is odd, $G$ has a total coloring $\pi$ which uses $m$ colors, $1, \ldots, m$. Now, define an edge-coloring $c$ of $K_{2m}$ as follows:

(a) For each edge $a_ia_k \in E(G)$, let $c(a_ia_k) = \pi(a_ia_k)$.
(b) For each edge $b jb_k \in E(H)$, let $c(b jb_k) = \pi(a_ia_k)$.
(c) For each edge $a.ib_i, 1 \leq i \leq m$, let $c(a_i b_i) = \pi(a_i)$.
(d) For each edge $a_i b_j, j \neq k$, let $c(a_i b_k) = [k - j]_m + m$.

Clearly, $c$ is a proper $(2m - 1)$-edge-coloring of $K_{2m}$. It is left to decompose $K_{2m}$ into $m$ multicolored isomorphic spanning trees. First, for each $i \in \{1, \ldots, m\}$, let $T_i$ be defined on the set $A \cup B$ and $E(T_i) = \{a_ia_{i+2t-1}, b_ib_{i+2t-1}, a_{i+1}b_{i+2t}\}$, $a_{i+1}b_{i+2t} \in E$ for $1 \leq i \leq m - 2$ and $T_i$ is a multicolored spanning tree, and all the $T_i$'s are isomorphic.

Now, if $m$ is not an odd integer, then $2m = 2m'$ where $t \geq 2$ and $m'$ is odd. In the case where $m' = 1$, $t$ must be at least 3. Then it is a direct consequence of Theorem A. Assume $m' \geq 3$. Thus $K_{2m'}$ admits an MTP by Theorem A except when $m' = 3$ and $t = 2$. Since this case can be handled by Lemma 2, we conclude the proof. 

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REFERENCES