THE OPTIMAL PEBBLING NUMBER OF THE CATERPILLAR

Chin-Lin Shiue and Hung-Lin Fu

Abstract. Let $G$ be a simple graph. If we place $p$ pebbles on the vertices of $G$, then a pebbling move is taking two pebbles off one vertex and then placing one on an adjacent vertex. The optimal pebbling number of $G$, $f'(G)$, is the least positive integer $p$ such that $p$ pebbles are placed suitably on vertices of $G$ and for any target vertex $v$ of $G$, we can move one pebble to $v$ by a sequence of pebbling moves. In this paper, we find the optimal pebbling number of the caterpillars.

1. INTRODUCTION

Suppose $p$ pebbles are distributed onto the vertices of $G$; then we have a so-called distribution $\delta$ where we let $\delta(v)$ be the number of pebbles distributed to $v \in V(G)$ and $\delta(H) = \sum_{v \in V(H)} \delta(v)$ for each induced subgraph $H$ of $G$. Note that now $\delta(G) = p$.

A pebbling move consists of moving two pebbles from one vertex and then placing one pebble at an adjacent vertex. If a distribution $\delta$ let us move, for any vertex $v$, at least one pebble to $v$ by applying pebbling moves repeatedly (if necessary), then $\delta$ is called a pebbling of $G$. Suppose $\delta$ is a distribution of $G$ and $H$ is an induced subgraph of $G$ and $v$ is a vertex in $H$. Let $\delta(H, v)$ denote the maximum number of pebbles which can be moved to $v$ by applying pebbling moves on $H$. Therefore, for each $v \in V(G)$ $\delta(G, v) > 0$ if $\delta$ is a pebbling of $G$. The pebbling number $f(G)$ of a graph $G$ is defined as the minimum number of pebbles $p$ such that any distribution with $p$ pebbles is a pebbling of $G$. The problem of pebbling graph was first proposed by M. Saks and J. Lagarias [1] as a tool for solving a number theoretic problem posed by Lemke and Kleitman [6], and some excellent results have been obtained, see [1, 2, 3, 5, 8, 12].

Received November 7, 2006, accepted August 23, 2007.
Communicated by Xu-Ding Zhu.
2000 Mathematics Subject Classification: 05C05.
Key words and phrases: Optimal pebbling, Caterpillar.
Motivated by the study of pebbling number, the notion of optimal pebbling was introduced later by L. Pachter et al. [10]. The optimal pebbling number of $G$, $f'(G)$, is $\min\{\delta(G) | \delta$ is a pebbling of $G\}$, and a distribution $\delta$ is an optimal pebbling of $G$ if $\delta$ is a pebbling of $G$ such that $\delta(G) = f'(G)$. First, L. Pachter et al. found the optimal pebbling number for the path.

**Theorem 1.1.** [10]. Let $P$ be a path of order $3t+r$, i.e., $|V(P)| = 3t + r$, where $0 \leq r < 3$. Then $f'(P) = 2t + r$. Later, several results have been obtained.

**Theorem 1.2.** [11]. $f'(C_n) = f'(P_n)$.

**Theorem 1.3.** [11]. For any graphs $G$ and $H$, $f'(G \times H) \leq f'(G)f'(H)$.

**Theorem 1.4.** [9] $f'(Q_n) = \left(\frac{4}{3}\right)n + O(\log n)$. Besides, Fu and Shiue devised a polynomial algorithm to determine the optimal pebbling number of the complete $m$-ary tree [4]. With regard to the complexity of determining the pebbling number, Milan and Clark showed that deciding whether $f'(G) \leq k$ for a graph $G$ and an integer $k$ is NP-complete [7]. This says that to determine the optimal pebbling number for a graph is difficult.

In order to find the optimal pebbling number of the caterpillar, we introduce the notion of weighted pebbling. Let $\alpha$ be a weighted function mapping from $V(G)$ into the set of positive integers. If $\delta$ is a distribution of $G$ and $\delta(G, v) \geq \alpha(v)$ for every vertex $v$, then $\delta$ is called an $\alpha$-weighted pebbling of $G$. In what follows, we call $\alpha$ a pebbling type of $G$ and the optimal $\alpha$-weighted pebbling number of $G$, $f'_\alpha(G)$, is $\min\{\delta(G) | \delta$ is an $\alpha$-weighted pebbling of $G\}$. Clearly, if $\alpha(v) = 1$ for each $v \in V(G)$, then $f'_\alpha(G) = f'(G)$. In this paper, we shall determine the optimal pebbling number for a graph is difficult.

2. **Main Result**

A tree $T$ is called a caterpillar if the deletion of all pendent vertices of the tree results in a path $P'$. For convenience, we shall call a path $P$ with maximum length which contains $P'$ a body of the caterpillar, and all the edges which are incident to pendent vertices are the legs of the caterpillar $T$. Furthermore, the vertex $v \in V(P)$ is a joint of $T$ provided that $\deg_T(v) \geq 3$ or $v$ is adjacent to the end vertices, see Figure 1 for an example.

Now, we are ready to prove the first lemma. First, we need the following facts. Since they are easy to see, we omit their proofs

**Fact 1.** Let $T$ be a tree and $\delta$ be a distribution of $T$, and let $v$ be a pendent vertex of $T$ which is adjacent to $u$. If $\delta^*$ is a distribution of $T - v$ defined by
\[
\delta^*(u) = \delta(u) + \left\lfloor \frac{\delta(v)}{2} \right\rfloor \quad \text{and} \quad \delta^*(w) = \delta(w) \quad \text{for each} \ w \in V(T) - \{u, v\}, \text{then} \\
\delta^*(T - v, w) = \delta(T, w) \quad \text{for each} \ w \in V(T) - \{v\}.
\]

**Fig. 1.** A caterpillar with 5 joints.

**Fact 2.** Let \( G \) be a graph, and let \( \delta_1 \) and \( \delta_2 \) are two distributions of \( G \). If \( \delta_1(v) \geq \delta_2(v) \) for each \( v \in V(G) \), then \( \delta_1(G, v) \geq \delta_2(G, v) \) for each \( v \in V(G) \).

The following lemma provides a recursive step to show that the problem of finding the optimal pebbling number of a caterpillar \( T \) is equivalent to the problem of finding the optimal \( \alpha \)-weighted pebbling number of a body of \( T \) for some type \( \alpha \).

**Lemma 2.1.** Let \( T \) be a tree, and let \( v \) be a pendant vertex of \( T \) which is adjacent to \( u \). If \( \alpha \) is a pebbling type of \( T \) satisfying \( \alpha(v) = 1 \) and \( \beta \) is a pebbling type of \( T - v \), defined by \( \beta(u) = \max \{1, \alpha(u)\} \) and \( \beta(w) = \alpha(w) \) for \( w \in V(T) - \{u, v\} \) then \( f^*_\alpha(T) = f^*_\beta(T - v) \).

**Proof.** Let \( \delta^* \) be an optimal \( \beta \)-weighted pebbling of \( T - v \). Then we choose a distribution \( \delta \) of \( T \) such that \( \delta(u) = 0 \) and \( \delta(w) = \delta^*(w) \) for each \( w \in V(T) - \{v\} \). Clearly, \( \delta(T) = \delta^*(T - v) = f^*_\beta(T - v) \) and \( \delta(T - v, w) = \delta^*(T - v, w) \geq \beta(w) = \alpha(w) \) for each \( w \in V(T) - \{v\} \). Since \( \delta(T - v, u) = \delta^*(T - v, u) \geq \beta(u) \geq 2 \), we have \( \delta(T, v) = \delta(v) + \left\lfloor \frac{\delta(T - v, w)}{2} \right\rfloor \geq \left\lfloor \frac{\beta(u)}{2} \right\rfloor \geq 1 = \alpha(v) \). This implies that \( \delta \) is an \( \alpha \)-weighted pebbling of \( T \) and it follows that \( f^*_\alpha(T) \leq f^*_\beta(T - v) \).

To prove that \( f^*_\alpha(T) \geq f^*_\beta(T - v) \), let \( \delta \) be an optimal \( \alpha \)-weighted pebbling of \( T \) and \( \delta^* \) be a distribution of \( T - v \) such that \( \delta^*(u) = \delta(u) + \delta(v) \) and \( \delta^*(w) = \delta(w) \) for each \( w \in V(T - v) - \{u\} \). Clearly, \( \delta^*(T - v) = \delta(T) = f^*_\alpha(T) \). By Fact 1 and Fact 2, we have

\[
\delta^*(T - v, w) \geq \delta(T, w) \geq \alpha(w) \quad \text{for each} \ w \in V(T - v).
\]

Since \( \beta(w) = \alpha(w) \) for each \( w \in V(T - v) - \{u\} \), \( \delta^* \) will be a \( \beta \)-weighted pebbling as long as \( \delta^*(T - v, u) \geq \beta(u) \).

First, if \( \alpha(u) \geq 2 \) then \( \beta(u) = \alpha(u) \). By \( (*) \), \( \delta^*(T - v, u) \geq \delta(T, u) \geq \alpha(u) = \beta(u) \). On the other hand, \( \alpha(u) \) must be 1 and \( \beta(u) = 2 \) by the hypothesis. By the definition of \( \delta^* \), if \( \delta(v) \geq 2 \) or \( \delta(T - v, u) \geq 2 \) then \( \delta^*(T - v, u) = \delta(T - v, u) + \delta(v) \geq 2 = \beta(u) \). Now, if \( \delta(T - v, u) = 0 \), then there are at least two pebbles on \( v \) in the distribution \( \delta \), i.e., \( \delta(v) \geq 2 \). Finally, if \( \delta(T - v, u) = 1 \)
then $\delta(v) \geq 1$. For otherwise, there is no way to move a pebble to $v$ by using the distribution $\delta$ and pebbling moves. Again, by the definition of $\delta^*$, we have $\delta^*(T - v, u) = \delta(T - v, u) + \delta(v) \geq 2 = \beta(u)$. This concludes the proof.

**Proposition 2.2.** Let $T$ be a caterpillar of order $n \geq 3$ and $P$ be a body of $T$. If $\alpha$ is a pebbling type of $P$ defined by $\alpha(v) = 2$ provided that $v$ is a joint of $T$ and $\alpha(v) = 1$ otherwise. Then $f'(T) = f'_\alpha(P)$.

**Proof.** This is a direct consequence of Lemma 2.1, by adding legs to $P$ repeatedly.

In what follows, we try to find an explicit formula for finding the optimal $\alpha$-weighted pebbling of a path $P$ with $\alpha(v) = 1$ or 2 for each $v \in V(P)$. Therefore, the optimal pebbling of a caterpillar can be obtained accordingly.

Throughout the rest of this paper, $T$ is a caterpillar, $P$ is a body of $T$ and $\alpha$ is a pebbling type of $P$ which is defined as in Proposition 2.2. Moreover, we let $S_1 = \{v \in V(P)|\delta(v) = 0 \text{ and } \delta(P, v) = 1\}$ where $\delta$ is a distribution of $P$. Then the following fact is obvious.

**Fact 3.** Let $\delta$ be a distribution of $P$. If $v \in S_1$ then there exists exactly one vertex $u$ adjacent to $v$ which satisfies the inequalities $2 \leq \delta(P, u) \leq 3$.

We start with finding a good lower bound for $f'_\alpha(P)$.

**Proposition 2.3.** If $\delta$ is an $\alpha$-weighted pebbling of $P$, then $\delta(P) \geq |V(P)| - \lfloor \frac{1}{2}|S_1|\rfloor$.

**Proof.** Since $\alpha(v) \geq 1$, it is easy to see that the lemma holds for $|V(P)| \leq 3$. Let $\delta$ be an $\alpha$-weighted pebbling of $P$, and let $S_0 = \{v \in V(P)|\delta(v) = 0\}$. Note that $\delta(P, v) \geq 2$ for each $v \in S_0 - S_1$. Hence, we have

$$
\sum_{v \in S_0} \delta(P, v) \geq |S_1| + 2|S_0 - S_1| = 2|S_0| - S_1.
$$

(1)

Let $P = v_1v_2 \cdots v_n$ where $n > 3$. For $i = 1, 2, \cdots, n$, we define the subpath $L_i = v_1v_2 \cdots v_i$ and the subpath $R_i = v_i v_{i+1} \cdots v_n$. For convenience, we denote $\delta(L_i, v_i)$ by $\ell(v_i)$ and $\delta(R_i, v_i)$ by $r(v_i)$ for $1 \leq i \leq n$. Then it is easy to see that $\ell(v_1) = \delta(v_1)$, $r(v_n) = \delta(v_n)$, $\ell(v_i) = \delta(v_i) + \lfloor \frac{\ell(v_{i+1})}{2} \rfloor$ for $2 \leq i \leq n$, and $r(v_i) = \delta(v_i) + \lfloor \frac{r(v_{i+1})}{2} \rfloor$ for $1 \leq i \leq n - 1$. This implies that $\delta(P, v) = \ell(v) + r(v)$ if $\delta(v) = 0$. So, we have

$$
\sum_{v \in S_0} \delta(P, v) = \sum_{v \in S_0} \ell(v) + \sum_{v \in S_0} r(v).
$$
Now, we will prove that \( \sum_{v \in S_0} \ell(v) \leq \delta(P) - |V(P)| + |S_0| \). Let \( s \) be a positive number. We define \( \phi_0(s) = s \) and \( \phi_i(s) = \left\lfloor \frac{\phi_{i-1}(s)}{2} \right\rfloor \) for each positive integer \( i \).

Clearly, \( \sum_{i=1}^{t} \phi_i(s) \leq s - 1 \) for any positive integer \( t \).

Consider the subpath \( P'' = v_{k+1}v_{k+2} \cdots v_{k+\ell}v_{k+\ell+1} \cdots v_{k+\ell + m} \), which satifies that \( \delta(v_{k+i}) \geq 1 \) for \( 1 \leq i \leq \ell \) and \( v_{k+\ell+j} \in S_0 \) for \( 1 \leq j \leq m \). Since \( \delta(v_{k+\ell+j}) = 0 \) for \( 1 \leq j \leq m \), we have

\[
\ell(v_{k+\ell+j}) = \phi_j(\ell(v_{k+\ell})) \text{ for } 1 \leq j \leq m.
\]

By (2), we also have

\[
\ell(v_{k+j}) = \phi_1(\ell(v_{k+j-1}))+\delta(v_{k+j}) \leq \ell(v_{k+j-1})-1+\delta(v_{k+j}) \text{ for } 2 \leq j \leq \ell.
\]

By (3), we have

\[
\phi_{m+1}(\ell(v_{k+\ell})) = \phi_1(\phi_m(\ell(v_{k+\ell}))) = \phi_1(\ell(v_{k+\ell+m})).
\]

By combining (2) and (4), we obtain

\[
\sum_{j=1}^{m+1} \phi_j(\ell(v_{k+\ell})) \leq \ell(v_{k+\ell}) - 1
\]

\[
\leq \ell(v_{k+\ell-1}) - 1 + \delta(v_{k+\ell}) - 1
\]

\[
= \ell(v_{k+\ell-2}) - 1 + \delta(v_{k+\ell-1}) - 1 + \delta(v_{k+\ell}) - 1
\]

\[
\vdots
\]

\[
\leq (\ell(v_{k+1}) - 1) + \sum_{j=2}^{\ell} (\delta(v_{k+j}) - 1).
\]

Also, by Combining (3) and (5), we have

\[
\sum_{v \in V(P'') \cap S_0} \ell(v) = \sum_{j=1}^{m} \ell(v_{k+\ell+j})
\]

\[
= \sum_{j=1}^{m} \phi_j(\ell(v_{k+\ell})) = \sum_{i=1}^{m+1} \phi_j(\ell(v_{k+\ell})) - \phi_{m+1}(\ell(v_{k+\ell}))
\]

\[
\leq (\ell(v_{k+1}) - 1) - \phi_1(\ell(v_{k+\ell+m})) + \sum_{j=2}^{\ell} (\delta(v_{k+j}) - 1).
\]
If \( k = 0 \) then \( \ell(v_{k+1}) = \ell(v_1) = \delta(v_1) = \delta(v_{k+1}) \), and so we have

\[
\sum_{v \in V(P_1') \cap S_0} \ell(v) \leq -\phi_1(\ell(u_1)) + \sum_{v \in V(P_1')} (\delta(v) - 1) = -\phi_1(\ell(u_1)) + \sum_{v \in V(P_1') \cap S_0} (\delta(v) - 1).
\]

Otherwise, \( k \geq 1 \) then \( \ell(v_{k+1}) = \phi_1(\ell(v_k) + \delta(v_{k+1})) \). This implies that

\[
\sum_{v \in V(P_1') \cap S_0} \ell(v) = \phi_1(\ell(v_k)) - \phi_1(\ell(v_{k+1})) + \sum_{v \in V(P_1') \cap S_0} (\delta(v) - 1). \quad (6')
\]

Let \( P = P_0 \sim P_1' \sim P_1 \sim P_2' \sim \cdots \sim P_{m-1}' \sim P_m' \sim P_m \) where \( P_i' \) is the maximal subpath such that for each vertex \( v \in V(P_i') \), \( \delta(v) \geq 1 \) for \( 1 \leq i \leq m \).

Note that \( P_0 = \emptyset \) if \( v_1 \in V(P_1') \) and \( P_m = \emptyset \) if \( v_m \in V(P_m') \). We also let \( u_i \) be the rightmost vertex of \( P_i \) for \( 1 \leq i \leq m \). Obviously, if \( P_0 \neq \emptyset \), then we have

(i) \( \ell(u_0) = 0 \) and \( \sum_{v \in V(P_0)} \ell(v) = 0 \);

and by combining (6), (6') and (i), we obtain

(ii) \[
\sum_{v \in V(P_1')} \ell(v) \leq -\phi_1(\ell(u_1)) + \sum_{v \in V(P_1')} (\delta(v) - 1) \quad \text{and} \quad \sum_{v \in V(P_1')} \ell(v) \leq \phi_1(\ell(u_{i-1})) - \phi_1(\ell(u_i)) + \sum_{v \in V(P_1')} (\delta(v) - 1) \quad \text{for} \quad 2 \leq i \leq m - 1; \quad \text{and}
\]

(iii) if \( P_m = \emptyset \) then \( \sum_{v \in V(P_m')} \ell(v) = 0 \leq \phi_1(\ell(u_{m-1})) + \sum_{v \in V(P_m')} (\delta(v) - 1) \).

Otherwise, \( P_m \neq \emptyset \) which implies

\[
\sum_{v \in V(P_m')} \ell(v) \leq \phi_1(\ell(u_{m-1})) + \sum_{v \in V(P_m')} (\delta(v) - 1) \quad \leq \phi_1(\ell(u_{m-1})) + \sum_{v \in V(P_m')} (\delta(v) - 1).
\]

By combining (i), (ii) and (iii), we have

\[
\sum_{v \in S_0} \ell(v) = \sum_{i=0}^{m} \sum_{v \in V(P_i')} \ell(v) \leq \sum_{i=1}^{m} \sum_{v \in V(P_i')} (\delta(v) - 1).
\]

Since \( \bigcup_{i=1}^{m} V(P_i') = V(P) - S_0 \) and \( \delta(P) = \sum_{v \in V(P) - S_0} \delta(v) \), we obtain
\[
\sum_{v \in S_0} \ell(v) \leq \sum_{v \in V(P)-S_0} (\delta(v) - 1) = \sum_{v \in V(P)-S_0} \delta(v) - (|V(P)| - |S_0|)
\]
\[
= \delta(P) - |V(P)| + |S_0|.
\]
By a similar argument as above, we also have \(\sum_{v \in S_0} r(v) \leq \delta(P) - |V(P)| + |S_0|\).
Hence,
\[
\sum_{v \in S_0} \delta(P, v) = \sum_{v \in S_0} \ell(v) + \sum_{v \in S_0} r(v) \leq 2(\delta(P) - |V(P)| + |S_0|).
\] (7)
By combining (1) and (7), we have
\[
2|S_0| - |S_1| \leq \sum_{v \in S_0} \delta(P, v) \leq 2(\delta(P) - |V(P)| + |S_0|),
\]
and the proof is complete.

In order to determine \(f'_{\alpha}(P)\), we also need the following notions. A subpath \(Q\) of \(P\) is said to be 1-maximal with respect to \(\alpha\), if \(Q\) is a maximal connected subgraph of \(P\) such that for each \(v \in V(Q)\), \(\alpha(v) = 1\) and for each vertex \(u\) which is adjacent to \(v\), \(\alpha(u) = 1\); and \(Q\) is 2-maximal with respect to \(\alpha\), if \(Q\) is a maximal connected subgraph of \(P\) such that for each adjacent pair \(u\) and \(w\) in \(V(Q)\), \(\alpha(u) = 1\) implies \(\alpha(w) = 2\) or \(\alpha(u) = \alpha(w) = 2\). For clearness, we give an example in Figure 2.

\[\begin{array}{cccc}
2\text{-maximal} & 1\text{-maximal} & 2\text{-maximal} & 2\text{-maximal} \\
(1 & 2 & 1 & 2) & (1 & 1 & 1 & 1) & (1 & 2 & 1 & 2) & (1 & 2 & 2 & 1)
\end{array}\]

Fig. 2. \(\alpha(P)\).

The following result can be derived from Theorem 1.1.

**Lemma 2.4.** Let \(Q\) be a 1-maximal subpath of \(P\) with respective to \(\alpha\). Then \(f'_{\alpha}(Q) = |V(Q)| - \left\lfloor \frac{|V(Q)|}{3} \right\rfloor\).

**Lemma 2.5.** Let \(Q = v_1v_2\cdots v_k, k \geq 3\) be a 2-maximal subpath of \(P\) with respective to \(\alpha\). Then \(f'_{\alpha}(Q) = k - 1\).

**Proof.** Let \(\delta\) be an optimal \(\alpha\)-weighted pebbling of \(P\). Then by Fact 3, \(v_i \notin S_1\) for \(2 \leq i \leq k - 1\). This implies that \(|S_1| \leq 2\). By Proposition 2.3, we have \(f'_{\alpha}(P) \geq k - 1\). Now, by letting \(\delta\) be the distribution satisfying \(\delta(v_1) = \delta(v_k) = 0, \delta(v_2) = 2\) and \(\delta(v_i) = 1\) for \(3 \leq i \leq k - 1\), we have \(f'_{\alpha}(P) \leq k - 1\). This concludes the proof.
Lemma 2.6. Let $P = v_1v_2 \cdots v_n$ and $Q = v_iv_{i+1} \cdots v_{i+k+1}$ be a subpath of $P$ where $2 \leq i < i + k + 1 \leq n - 1$, and let $\delta$ be a $\alpha$-weighted pebbling of $P$. If $\alpha(v_{i-1}) = \alpha(v_{i+k+2}) = 2$ and $\alpha(v_{i+j}) = 1$ for $j = 0, 2, \cdots, k + 1$, then $|V(Q) \cap S_1| \leq 2[\frac{k}{3}] + 2$.

Proof. Let $S_2 = \{v \in V(Q) | \delta(P, v) \geq 2\}$ and $|S_2| = x$. By the hypothesis and Fact 3, we conclude that

$$\text{if } v_{i+1} \in S_2 \text{ then } v_i \notin S_1 \text{ and if } v_{i+k} \in S_2 \text{ then } v_{i+k+1} \notin S_1. \quad (A)$$

By Fact 3, each vertex $v$ in $(V(Q) \setminus \{v_i, v_{i+k+1}\}) \cap S_1$ is adjacent to exactly one vertex in $S_2$. Also, since $Q$ is a subpath, each vertex in $S_2$ is adjacent to at most two vertices in $(V(Q) \setminus \{v_i, v_{i+k+1}\}) \cap S_1$. Hence, we have $|V(Q) \cap S_1| \leq 2x + 2$. This implies

$$x \geq \frac{|V(Q) \cap S_1|}{2} - 1. \quad (B)$$

Since $V(Q) \cap S_1 \subseteq V(Q) - S_2$, we have

$$|V(Q) \cap S_1| \leq |V(Q)| - |S_2| = |V(Q)| - x \leq k + 3 - |V(Q) \cap S_1|/2.$$

It follows that $|V(Q) \cap S_1| \leq \frac{2k}{3} + 2$. Let $k = 3t + r$ and $0 \leq r \leq 2$. Then $t = \lfloor \frac{k}{3} \rfloor$.

Case 1. $r = 0$ or 1. We have $|V(Q) \cap S_1| \leq 2t + \left\lceil \frac{2r}{3} \right\rceil + 2 = 2t + 2 = 2\lfloor \frac{k}{3} \rfloor + 2$.

Case 2. $r = 2$. We have $|V(Q) \cap S_1| \leq 2t + \left\lceil \frac{2r}{3} \right\rceil + 2 = 2t + 3$. Suppose that $|V(Q) \cap S_1| = 2t + 3$. Then $x \leq |V(Q)| - |V(Q) \cap S_1| = (3t + 2 + 2) - (2t + 3) = t + 1$. But, we have $x \geq (2t + 3)/2 - 1 = t + 1/2$ by (B) and thus $x = t + 1$. This implies that $|S_2| + |V(Q) \cap S_1| = 3t + 4 = 2k = |V(Q)|$. Therefore, for each $v \in V(Q)$, either $v \in S_1$ or $v \in S_2$. By (A), Fact 3 and the hypothesis, if $v_{i+j} \in S_2$ and $v_{i+j+1} \in S_1$ then $v_{i+j+2} \in S_1$, and if $v_{i+j} \in S_1$ and $v_{i+j+1} \in S_1$ then $v_{i+j+2} \in S_2$ for $0 \leq j \leq k - 1$. This implies that $|V(Q) \cap S_1|$ is even. It is a contradiction to our assumption. Therefore, we have $|V(Q) \cap S_1| \leq 2t + 2 = 2\lfloor \frac{k}{3} \rfloor + 2$.

Now, we are ready for the main theorem.

Theorem 2.7. Let $T$ be a caterpillar with $P$ a body of $T$ and $|V(P)| = n \geq 3$. Let $\alpha(v) = 2$ if $v$ is a joint of $T$ and $\alpha(v) = 1$ otherwise. Let $P'_1, P'_2, \cdots, P'_m$ be 2-maximal subpaths of $P$ with respect to $\alpha$ and $P_i$ be a subpath between $P'_i$ and $P'_{i+1}$ for $i = 1, 2, \cdots, m - 1$. Then $f'(T) = n - m - \sum_{i=1}^{m-1} \left\lceil \frac{|V(P_i)|}{3} \right\rceil$. 


Therefore, we obtain
\[
\text{Hence, we have}
\]
\[
1 \leq \sum_{i=1}^{m-1} \left\lfloor \frac{|V(P_i)|}{3} \right\rfloor.
\]

By Lemma 2.4 and Lemma 2.5, it is easy to see
\[
\text{with respect to each } P_i \text{ and } u_i \text{ respectively. Note that } w_i \text{ is an empty graph or a 1-maximal subpath of } P_i \text{ with respect to } \alpha. \text{ If } P_i \text{ is an empty graph then } |V(P_i)| - \left\lfloor \frac{|V(P_i)|}{3} \right\rfloor = 0. \text{ Combining with Lemma 2.4 and Lemma 2.5, it is easy to see}
\]
\[
f'_n(P) \leq \sum_{i=1}^{m} (|V(P_i')| - 1) + \sum_{i=1}^{m-1} \left( |V(P_i)| - \left\lfloor \frac{|V(P_i)|}{3} \right\rfloor \right)
\]
\[
= \sum_{i=1}^{m} |V(P_i')| + \sum_{i=1}^{m-1} |V(P_i)| - m - \sum_{i=1}^{m-1} \left\lfloor \frac{|V(P_i)|}{3} \right\rfloor
\]
\[
= n - m - \sum_{i=1}^{m-1} \left\lfloor \frac{|V(P_i)|}{3} \right\rfloor.
\]

Therefore, it is left to show that the above upper bound is also a lower bound. By Proposition 2.3, we have
\[
f'_n(P) \geq |V(P)| - \left\lfloor \frac{1}{2} |S_1| \right\rfloor = n - \left\lfloor \frac{1}{2} |S_1| \right\rfloor.
\]

Hence, it suffices to show
\[
|S_1| \leq 2m + \sum_{i=1}^{m-1} \left\lfloor \frac{|V(P_i)|}{3} \right\rfloor.
\]

For \(1 \leq i \leq m\), let \(u_i\) and \(w_i\) be the leftmost vertex and the rightmost vertex of \(P_i\) respectively. Note that \(w_i\) and \(u_{i+1}\) are adjacent to the left end vertex and the right end vertex of \(P_i\) respectively. We denote the subpath induced by \(\{w_i, u_{i+1}\} \cup V(P_i)\) by \(w_i \sim P_i \sim u_{i+1}\). By Fact 3, we have \(V(P_i') - \{u_i, w_i\} \cap S_1 = \emptyset\) for \(1 \leq i \leq m\). This implies that
\[
S_1 = V(P) \cap S_1 = \{u_1, w_m\} \cap S_1 \cup \bigcup_{i=1}^{m-1} [V(w_i \sim P_i \sim u_{i+1}) \cap S_1].
\]

Hence, we have
\[
|S_1| = |\{u_1, w_m\}| + \sum_{i=1}^{m-1} |V(w_i \sim P_i \sim u_{i+1}) \cap S_1|.
\]

By Lemma 2.6, we have
\[
|V(w_i \sim P_i \sim u_{i+1}) \cap S_1| \leq 2 \left\lfloor \frac{|V(P_i)|}{3} \right\rfloor + 2 \text{ for } 1 \leq i \leq m - 1.
\]

Therefore, we obtain
\[
|S_1| \leq 2 + \sum_{i=1}^{m-1} (2 \left\lfloor \frac{|V(P_i)|}{3} \right\rfloor + 2) = 2m + \sum_{i=1}^{m-1} 2 \left\lfloor \frac{|V(P_i)|}{3} \right\rfloor.
\]
This concludes the proof.

Before we finish this paper, we give an example to clarify the idea used in this paper.

Example. Let $T$ be a caterpillar in Figure 3. Here, $n = 25$, $m = 4$, $\sum_{i=1}^{m-1} \left\lfloor \frac{|V(P_i)|}{3} \right\rfloor = 2$ and $f'(T) = 25 - 4 - 2 = 19$.

![Fig. 3. An optimal $\alpha$-weighted pebbling of $P$.](image)

ACKNOWLEDGMENT

The authors would like to thank the referees for their helpful comments in revising this paper.

REFERENCES


Chin-Lin Shiue  
Department of Applied Mathematics,  
Chung Yuan Christian University,  
Chung Li,  
Taiwan  
E-mail: clshiue@math.cycu.edu.tw

Hung-Lin Fu  
Department of Applied Mathematics,  
National Chiao Thung University,  
Hsin Chu,  
Taiwan  
E-mail: hlfu@math.nctu.edu.tw