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Maximal Configurations of Stars

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Abstract. A star $S_q$ with $q$ edges is a complete bipartite graph $K_{1,q}$. Two figures of the complete graph $K_n$ on a given set of $n$ vertices are compatible if they are edge-disjoint, and a configuration is a set of pairwise compatible figures. In this paper, we take stars as our figures. A configuration $C$ is said to be maximal if there is no figure (star) $f \not\in C$ such that $\{f\} \cup C$ is also a configuration. The size of a configuration $F$, denoted by $|F|$, is the number of its figures.

Let $\text{Spec}(n,q)$ (or simply $\text{Spec}(n)$) denote the set of all sizes such that there exists a maximal configuration of stars with this size. In this paper, we completely determine $\text{Spec}(n)$, the spectrum of maximal configurations of stars. As a special case, when $n$ is an order of a star system, we obtain the spectrum of maximal partial star systems.

1. Introduction.

Let $S_q$ be a star on $q + 1$ vertices. A complete graph $K_n$ is said to have a $G$-decomposition $G[n]$ if it is a union of edge-disjoint subgraphs of $K_n$ each of which is isomorphic to a fixed graph $G$. The basic problem connected with the $G$-decomposition is to determine, for a given graph $G$, the necessary and sufficient conditions on $n$ for the existence of a decomposition $G[n]$. When the graph $G$ is itself a complete graph $K_k$, then the decomposition $K_k[n]$ is known as a balanced incomplete block design (BIBD) [1]. For the case where $G$ is a star, the problem is completely settled by M. Tarsi in a more general way [3]. As a special case, he proved that the necessary and sufficient conditions for the existence of an $S_q$-decomposition are that $n \geq 2q$ and $n(n - 1) \equiv 0 \pmod{2q}$. This decomposition may also be referred to an $(n,q)$ star system (or simply a star system). If we start with a fixed $q$, then for the number $n$ which does not satisfy the necessary and sufficient conditions mentioned above we can consider the problem of packing $K_n$ with as many stars, $S_q$ as possible.

Two figures of the complete graph $K_n$ on a given set of $n$ vertices are compatible if they are edge-disjoint, and a configuration is a set of pairwise compatible figures. In what follows, our configuration will be a set of pairwise edge-disjoint stars of the complete graph $K_n$. A configuration $C$ is said to be maximal if there is no star $f \not\in C$ such that $\{f\} \cup C$ is also a configuration. The size of a configuration $F$, denoted by $|F|$, is the number of its stars. Obviously, if $|F| = \left(\binom{n}{2}/q\right)$, then $F$

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is maximal. We are also interested in configurations which are maximal but have size less than \( \binom{n}{s} / q \). Thus, let \( \text{Spec}(n, q) \) or simply \( \text{Spec}(n) \) denote the set of all sizes such that there exists a maximal configuration of stars with this size. In this paper, we completely determine \( \text{Spec}(n) \), the spectrum of maximal configurations of stars. As a special case, when \( n \) is an order of a star system, we obtain the spectrum of maximal partial star systems.

For the rest of this paper, without mentioning otherwise, we consider configuration of \( K_n \), where \( n = mq + s, 1 \leq s \leq q \), in which the figures are stars. Since there is no star \( S_q \) in \( K_s \), we also assume \( m \geq 1 \). In what follows, we will use a \((q + 1) \times b\) array \( A = [a_{ij}] \) to represent a configuration with \( b \) stars where the first row represents the vertices of the centers and the degree one vertices of the \( j \)th star are \( a_{2j}, a_{3j}, \ldots, a_{(q+1)j} \). Figure 2.1 is an example of such a representation.

![Figure 2.1](image)

Let \( A = [a_{ij}] \) be an array which represents a configuration \( F \). Then, by observation, the degree of a vertex \( v \) in \( F \), \( \deg_F(v) \), can be obtained as \( \alpha q + \beta \) where \( \alpha \) is the number of \( a_{1j}, j = 1, 2, \ldots, b \), such that \( a_{1j} = v \) and \( \beta \) is the number of \( a_{ij} \) which is equal to \( v \), \( i = 2, 3, \ldots, q + 1 \) and \( j = 1, 2, \ldots, b \). Now, consider a configuration \( F \) on \( n \) vertices. If for each vertex \( v \), the degree of \( v \) in the graph \( K_n \setminus F \), the complement of \( F \), is less than \( q \), then \( F \) is a maximal configuration, that is, if \( \deg_F(v) \) is greater than \( n - 1 - q \), we conclude that \( F \) is a maximal configuration. Since, we will use this result often, we list it as a lemma.

**Lemma 1.1.** In a configuration \( F \), if \( \deg_F(v) \geq n - q \) for each \( v \in V(K_n) \), then \( F \) is maximal and the number of columns of an array which represents \( F \) is in \( \text{Spec}(n) \). Conversely, if \( F \) is a configuration in which there is a vertex of degree 1 less than \( n - q \), then \( F \) is not a maximal configuration.

**Corollary 1.2.** \( \min \text{Spec}(n) \geq \left\lceil \frac{n(n-q)}{2q} \right\rceil \).

What we try to prove in this paper is the following
Theorem 1.3. \( (i) \) \( \text{Spec}(n) = \{s, s+1, \ldots, 2s-1\} \) if \( n = q + s \) and \( 1 \leq s \leq q \).

\( (ii) \) \( \text{Spec}(n) = \left\{ x: \left\lfloor \frac{n - q}{2q} \right\rfloor \leq x \leq \left\lfloor \frac{n - 1}{2q} \right\rfloor \right\} \) if \( n \geq 2q + 1 \).

2. The main results.

We will start our construction with the smaller \( n \). (We note here that the maximal configuration of size \( \min \text{Spec}(n) \) is just the smallest possible value of our construction.)

First, consider \( n = q + s \). Let \( b = s + t, \delta = q - t \), and \( \gamma = \left\lfloor \frac{b - 1}{2} \right\rfloor \) where \( 1 \leq s \leq q \) and \( 0 \leq t \leq q \). Also, let \( V(K_n) = \{1, 2, 3, \ldots, b, v_1, v_2, \ldots, v_b\} \) and construct two arrays \( B \) and \( C \) as in Figure 2.2.

\[
B: \begin{bmatrix}
2 & 3 & 4 & \ldots & b & 1 \\
3 & 4 & 5 & \ldots & 1 & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\gamma + 1 & \gamma + 2 & \gamma + 3 & \ldots & \gamma - 1 & \gamma \\
\end{bmatrix}
\]

\[
C: \begin{bmatrix}
v_1 & v_1 & v_1 & \ldots & v_1 \\
v_2 & v_2 & v_2 & \ldots & v_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
v_\delta & v_\delta & v_\delta & \ldots & v_\delta \\
\end{bmatrix}
\]

Figure 2.2

Now we are ready to construct an array which represents a maximal configuration. If \( \left\lfloor \frac{6\delta}{b} \right\rfloor \geq q - \gamma \), then \( A_3 \) (Figure 2.3) can be defined as follows: \( A_3(1, j) = j, 1 \leq j \leq b; A_3(i, j) = v_{x+1} \) if \( b(i - 2) + j - 1 = sx + r \) for some \( 0 \leq r < s \), where \( 2 \leq i \leq q + 1, 1 \leq j \leq b \), and \( b + j \leq s\delta \); \( A_3(i, j) = B(i + \gamma - q - 1, j) \) if \( b + j > s\delta \).

\[
A_3: \begin{bmatrix}
1 & 2 & 3 & \ldots & v_1 & v_1 & v_1 & \ldots & v_2 & v_2 & v_2 & \ldots & v_3 & v_3 & \ldots \\
v_1 & v_1 & v_1 & \ldots & v_1 & v_2 & v_2 & \ldots & v_2 & v_3 & v_3 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
\vdots & \vdots & \vdots & \ddots & v_\delta & v_\delta & \ldots & v_\delta & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{bmatrix}
\]

Figure 2.3

For the situation \( \left\lfloor \frac{6\delta}{b} \right\rfloor < q - \gamma \), \( A_4 \) (Figure 2.4) can be defined similarly except that the whole of \( B \) is in the bottom of \( A_4 \); for each \( x \leq e, v_x \) occurs \( b' + 1 \) times and \( v_y \) occurs \( b' \) times whenever \( e < y \leq \delta \). (\( b' \) is a number between \( s \) and \( b \) such that \( (q - \gamma) b = \delta b' + e, 0 \leq e < \delta \)).

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It is not difficult to see that if $\delta \geq q - \gamma$, such an integer solution for $b'$ exists. Hence $q - t = \delta \geq q - \gamma$, that is, $t \leq \left\lfloor \frac{q+1}{2} \right\rfloor$. This implies that $t \leq s - 1$. Thus we can construct $A_4$ for each $0 \leq t \leq s - 1$, and in the corresponding configuration $F$, $\deg_P(v) \geq s, v \in V(K_{q+s})$, so by Lemma 2.1 we have a maximal configuration of stars whenever $0 \leq t \leq s - 1$. Equivalently, $\{s, s + 1, \ldots, 2s - 1\} \subseteq \text{Spec}(q + s)$. Now, suppose $x \in \text{Spec}(q + s)$, then the following two inequalities hold:

1. $xq \geq (q + s - x)s$, and
2. $xq \leq \frac{x(x - 1)}{2} + (q + s - x)x$.

It is easy to see $s \leq x \leq 2s - 1$. Hence we have the following

**Proposition 2.2.** For $1 \leq s \leq q$, $\text{Spec}(q + s) = \{s, s + 1, \ldots, 2s - 1\}$.

Next, if $n = 2q + s$, $1 \leq s \leq q$, and $\left\lfloor \frac{s(q+s)}{2q} \right\rfloor \leq t \leq q$. We note here that $q + s + \left\lfloor \frac{s(q+s)}{2q} \right\rfloor = \left\lfloor \frac{(2q+s)(q+s)}{2q} \right\rfloor = \min \text{Spec}(2q + s)$. Let $b = q + s + t$, $\delta = q - t$ and $\gamma = \left\lfloor \frac{b-1}{2} \right\rfloor$. $B$ and $C$ are defined as in Figure 2.2. We will construct $A_5$ in a similar way. First, if $\left\lfloor \frac{(q+s)b}{2q} \right\rfloor \geq q - \gamma$, $A_5$ (Figure 2.5) can be defined as follows: $A_5(1,j) = j, 1 \leq j \leq b$; $A_5(c,j) = v_{x+1}$ if $b(i-2) + j - 1 = (q + s)x + r$ for some $0 \leq r < q + s$ where $2 \leq i \leq q + 1, 1 \leq j \leq b$, and $bi + j \leq (q + s)\delta$; $A_5(i,j) = B(i + \gamma - q - 1, j)$ if $bi + j > (q + s)\delta$. 

![Figure 2.5](image-url)
Let $g$ be the number of rows in $A_5$ (Figure 2.5) which are below the rows containing the $v_i$'s, $i = 1, 2, \ldots, \delta$. If $g < s$, then

$$(q+s)\delta + s(q+s+t) = (q+s)\delta + (s-g)(q+s+t) + g(q+s+t) > g(q+s+t).$$

This implies that $t < \frac{s(q+s)}{2q} \leq \left[ \frac{s(q+s)}{2q} \right]$ which is a contradiction. Hence $g \geq s$, and we obtain a maximal configuration by Lemma 2.1. Secondly, if $\left\lfloor \frac{(q+s)\delta}{b} \right\rfloor < q - \gamma$, similar to the case $n = q + s$, the matrix is similar to $A_4$ except that $q + s < b' < b$. The fact that it represents a maximal configuration follows by a direct computation; we omit the details. So far, we have $\{q + s + \left\lfloor \frac{s(q+s)}{2q} \right\rfloor, q + s + \left\lfloor \frac{s(q+s)}{2q} \right\rfloor + 1, \ldots, 2q + s\} \subseteq \text{Spec} \ (2q + s)$. For the large values, we need other constructions. Let $b = 2q + s - y$, $1 \leq y \leq s - 1$. By the array in Figure 2.6, it is not difficult to see that $b + 2y \in \text{Spec} \ (2q + s)$, that is, $\{2q + s + 1, 2q + s + 2, \ldots, 2q + 2s - 1\} \subseteq \text{Spec} \ (2q + s)$.

$$A_6:\begin{array}{cccccc|cccc}
1 & 2 & 3 & \ldots & b-1 & b & v_1 & v_1 & v_2 & v_2 & \ldots & v_y & v_y \\
2 & 3 & 4 & \ldots & b & 1 & 1 & q+1 & 2q+1 \\
3 & 4 & 5 & \ldots & 1 & 2 & 2 & q+2 & 2q+2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
q+1 & q+2 & q+3 & \ldots & q-1 & q & q & 2q & \vdots \\
\end{array}$$

Figure 2.6

We still have some values left. Let us start with another array which represents a maximal configuration with size $2q + 2s - 1$. See Figure 2.7. (We omit the detail definition of $A_7$.)

$$A_7:\begin{array}{cccccc|cccc}
1 & 2 & 3 & \ldots & 2q-2 & 2q-1 & v_1 & v_2 & \ldots & v_s \\
2 & 3 & 4 & \ldots & 2q-1 & 1 & 1 & 1 & q+1 & q+1 & \ldots & q+1 \\
3 & 4 & 5 & \ldots & 1 & 2 & 2 & 2 & q+2 & q+2 & \ldots & q+2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
q & q+1 & q+2 & \ldots & q-2 & q-1 & q-1 & q-1 & 2q-1 & 2q-1 & \ldots & 2q-1 \\
2q & 2q & 2q & \ldots & 2q & 2q & q & q & 2q & 2q & \ldots & 2q \\
\end{array}$$

Figure 2.7

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As can be seen in the array, we have already used up the degree of the vertices 1, 2, \ldots, 2q, and we are going to adjust the degree of \(v_1, v_2, \ldots, v_s\) in order to obtain a maximal configuration with larger size. From \(A_7\), we have that in the representing configuration \(F\), \(\deg_F(v_i) = 2q, i = 1, 2, \ldots, s\). Thus, \(K_n \setminus F\) has \(\binom{s}{2}\) edges. If \(\binom{s}{2} < q\), then \(2q + 2s - 1 = \max \Spec(2q + s)\), we are done. If \(\binom{s}{2} \geq q\), we can adjust \(A_7\) by replacing the \(2q - 1\) edges \(\{q + 1, 1\}, \{q + 2, 1\}, \ldots, \{2q - 1, 1\}, \{v_1, 1\}, \{v_1, q + 1\}, \ldots, \{v_1, q + s - 2\}, \{v_2, q + s - 2 - 1\}, \ldots, \{v_2, q + 2(s - 2) + 1\}, \ldots, \{v_3, q + 2(s - 2) + 1\}, \ldots, \{v_3, q + 2(s - 2) + (s - 3)\}, \ldots\),

and \(\{v_i, q + \sum_{j=1}^i (s - j) + h - 1\}\) (in order) with \(\{q + 1, v_1\}, \{q + 2, v_1\}, \ldots, \{q + s - 2, v_1\}, \{q + s - 2 + 1, v_2\}, \ldots, \{2q - 1, v_i\}, \{v_1, v_2\}, \{v_1, v_3\}, \ldots, \{v_1, v_s\}, \{v_2, v_3\}, \ldots, \{v_2, v_s\}, \{v_3, v_4\}, \ldots\), and \(\{v_i, v_{i+h}\}\) respectively, where the first element of the edge represents the center of the star in which this edge belongs, and \(\sum_{j=1}^i (s - j) + h - 1 = q - 1\). In this way, we can add one more star with center 1 whose edges are \(\{1, q + 1\}, \{1, q + 2\}, \ldots, \{1, 2q - 1\}\) and \(\{1, v_1\}\). This implies that \(2q + 2s \in \Spec(2q + s)\). Now if \(\binom{s}{2} - q \geq q\), we can use a similar process to obtain a new star with center 2 except the new edges which we will use are those \(q\) edges \(\{v_i, v_{i+h+1}\}, \ldots, \{v_i, v_s\}, \{v_i, v_{i+2}\}, \ldots, \{v_i, v_{i+1}\}, \ldots\), \(\{v_j, v_{j+h'}\}\). (Replace \(\{1, 2\}, \{q + 2, 2\}, \{q + 3, 2\}, \ldots, \{2q - 1, 2\}, \{v_i, q + 2\}, \ldots\) and \(\{v_j, 2q - 1\}\) (in order) with \(\{1, v_{i+h+2}\}, \{q + 2, v_i\}, \ldots, \{2q - 1, v_i\}, \{v_i, v_{i+h+1}\}, \{v_i, v_{i+h+2}\}, \{v_i, v_{i+h+3}\}, \ldots, \{v_i, v_s\}, \ldots\), and \(\{v_j, v_{j+h'}\}\), respectively.) We will stop this process whenever the number of edges which are not used is less than \(q\). Since the same \(v_i\) will not occur in the same column which is not difficult to see, thus we have

Proposition 2.3. For \(1 \leq s \leq q\),

\[
\Spec(2q + s) = \left\{ q + s + \left\lfloor \frac{s(q + s)}{2q} \right\rfloor , \right. \\
q + s + \left\lfloor \frac{s(q + s)}{2q} \right\rfloor + 1, \ldots, 2q + 2s - 1 + \left\lfloor \frac{s}{2} \right\rfloor + \left\lfloor \frac{s}{q} \right\rfloor \\
\left\lfloor \frac{s(q + s)}{2q} \right\rfloor + 1, \ldots, 4q + 3s - 1 + \left\lfloor \frac{s}{2} \right\rfloor \right\} \subseteq \Spec(3q + s).
\]

On the case \(n = 3q + s\), let \(n' = 2q + s\). Construct a maximal configuration \(F\) on \(n'\) vertices. Add \(q\) new vertices and \(2q + s\) stars by joining each of the \(n'\) vertices to every new vertex. Then the new configuration is also maximal. Thus we have \(\min \Spec(3q + s) = \min \Spec(2q + s) + 2q + s\), and \(2q + s + x \in \Spec(3q + s)\) for each \(x \in \Spec(2q + s)\). By Proposition 2.3 we conclude that
If we start with a maximal configuration of size $4q + 3s - 1$, then we can follow a similar process as in the case $n = 2q + s$ to obtain $\Spec(3q + s)$. We will give the configuration of size $4q + 3s - 1$ and omit the details. (Figure 2.8.)

**Proposition 2.4.** For $1 \leq s \leq q$,

$$
\Spec(3q + s) = \left\{ 3q + 2s + \left\lfloor \frac{s(q+s)}{2q} \right\rfloor, 3q + 2s + \left\lfloor \frac{s(q+s)}{2q} \right\rfloor + 1, \ldots, \left\lfloor \frac{(3q+s)(3q+s-1)}{2q} \right\rfloor \right\}.
$$

\[
M_1 := \begin{bmatrix}
1 & 2 & \ldots & 2q - 1 & 2q + 1 & 2q + 2 & \ldots & 3q & 2q + 1 & 2q + 2 & \ldots & 3q \\
2 & 3 & \ldots & 1 & 2q & 2q & \ldots & 2q & q & q & \ldots & q \\
3 & 4 & \ldots & 2 & 1 & 1 & \ldots & 1 & q + 1 & q + 1 & \ldots & q + 1 \\
q & q + 1 & \ldots & q - 1 & q - 2 & q - 2 & \ldots & q - 2 & 2q - 2 & 2q - 2 & \ldots & 2q - 2 \\
2q & 2q & \ldots & 2q & q - 1 & q - 1 & \ldots & q - 1 & 2q - 1 & 2q - 1 & \ldots & 2q - 1 \\
v_1 & v_2 & \ldots & v_s & v_1 & v_2 & \ldots & v_s & v_1 & v_2 & \ldots & v_s \\
2q & 2q & \ldots & 2q & q & q & \ldots & q & 2q + 1 & 2q + 1 & \ldots & 2q + 1 \\
q - 2 & q - 2 & \ldots & q - 2 & 2q - 2 & 2q - 2 & \ldots & 2q - 2 & 3q - 1 & 3q - 1 & \ldots & 3q - 1 \\
q - 1 & q - 1 & \ldots & q - 1 & 2q - 1 & 2q - 1 & \ldots & 2q - 1 & 3q & 3q & \ldots & 3q
\end{bmatrix}
\]

$$
A_8 := [ M_1 | M_2 ]
$$

**Figure 2.8**

Now, we are ready to consider $n = mq + s, m \geq 4$. By a direct counting, we have

$$
\min \Spec(n) = \min \Spec(3q + s) + (3q + s)(m - 3) + \binom{m - 3}{2}q
$$

$$
= \min \Spec(2q + s) + (2q + s)(m - 2) + \binom{m - 2}{2}q.
$$

As a special case of $n = q + s$, $\Spec(2q) = \{q, q + 1, \ldots, 2q - 1\}$. If $m$ is even, then

$$
\Spec(n) \supseteq \left\{ x : x = y + (2q + s)(m - 2) + \binom{m - 2}{2}q - \frac{m - 2}{2} \cdot q + z_1 + z_2 + \ldots + z_\frac{m - 2}{2}, \right. \\
\left. \text{where } y \in \Spec(2q + s) \text{ and } z_i \in \Spec(2q), i = 1, 2, \ldots, \frac{m - 2}{2} \right\}.
$$

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This implies that

\[
Spec \supseteq \left\{ q + s + \left[ \frac{s(q + s)}{2q} \right] + (2q + s)(m - 2) + \binom{m-2}{2} q, \ldots, \right.
\]
\[
\left. \left[ \frac{(2q+s)(2q+s-1)}{2q} \right] + (2q+s)(m-2) + \binom{m-2}{2} q - \frac{m-2}{2} \cdot q + \frac{m-2}{2} (2q-1) \right\}.
\]

The last element of the set is equal to

\[
\left[ \frac{(mq + s)(mq + s - 1)}{2q} \right] = \max Spec (mq + s).
\]

For the situation \( m \) is odd and \( m > 3 \), we have

\[
Spec(n) \supseteq \left\{ x : x = y + (3q+s)(m-3) + \binom{m-3}{2} q - \frac{m-3}{2} \cdot q + z_1 + z_2 + \ldots + z_{\frac{m-3}{2}}, \right.
\]
\[
\left. \text{where } y \in Spec (3q + s) \text{ and } z_i \in Spec (2q), i = 1, 2, \ldots, \frac{m-3}{2} \right\}.
\]

This concludes this case.

**Proposition 2.5.** For \( 1 \leq s \leq q, m \geq 4 \),

\[
Spec (mq + s) = \left\{ q + s + \left[ \frac{s(q + s)}{2q} \right] + (2q + s)(m - 2) + \binom{m-2}{2} q, \right.
\]
\[
\ldots, \left[ \binom{mq + s}{2} / q \right] \right\}
\]
\[
= \left\{ \left[ \frac{(mq + s)(mq + s - q)}{2q} \right], \ldots, \left[ \binom{mq + s}{2} / q \right] \right\}
\]

Combining the above propositions, we have proved Theorem 1.3. As a special case when \( \binom{mq + s}{2} / q \) is an integer, we obtain a star system of order \( mq + s \) and also in this case we have the spectrum of partial maximal star systems.

### 3. Acknowledgement.

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References