A Special Partition of the Set $I_n$

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Abstract. Let $I_n$ be the set $\{1, 2, \ldots, n\}$, and $(s_1, s_2, \ldots, s_k)$ be a $k$-tuple of non-negative integers such that $\sum_{i=1}^k s_i = \binom{n+1}{2}$. We will say that $I_n$ can be partitioned into subsets of type $(s_1, s_2, \ldots, s_k)$ provided that there exists a collection of $k$ mutually disjoint subsets of $I_n$, $A_1, A_2, \ldots, A_k$ such that $\sum_{i=1}^k A_i = I_n$ and $\sum_{x \in A_i} x = s_i$ for each $i \in \{1, 2, \ldots, k\}$. In this paper, we study for which $k$-tuple $(t_1, t_2, \ldots, t_k)$ we can partition $I_n$ into subsets of type $(t_1, t_2, \ldots, t_k)$, and we are able to show that $I_n$ can be partitioned into subsets of type $(m, m+1, \ldots, m+k-2, \ell)$ where $m \geq 0$, $0 \leq \ell \leq \binom{n+1}{2}$, and $m(k-1)+\ell = \binom{n+1}{2}$. Furthermore, we conjecture that $I_n$ can be partitioned into subsets of type $(m_1, m_2, m_3, \ldots, m_k)$ provided that $\sum_{i=1}^k m_i = \binom{n+1}{2}$ and $m_i \geq n$ for each $i \in \{1, 2, \ldots, k-1\}$.

1. Introduction

Let $I_n$ be the set $\{1, 2, \ldots, n\}$, and $(s_1, s_2, \ldots, s_k)$ be a $k$-tuple of non-negative integers such that $\sum_{i=1}^k s_i = \binom{n+1}{2}$. We will say that $I_n$ can be partitioned into subsets of type $(s_1, s_2, \ldots, s_k)$ provided that there exists a collection of $k$ mutually disjoint subsets of $I_n$, $A_1, A_2, \ldots, A_k$, such that $\bigcup_{i=1}^k A_i = I_n$ and $\sum_{x \in A_i} x = s_i$ for each $i \in \{1, 2, \ldots, k\}$. Some results have been obtained in the study of this problem. We list them as propositions.

Proposition 1.1. [4] $I_n$ can be partitioned into subsets of type $(m, m, \ldots, m)$ ($k$-tuple) provided that $km = \binom{n+1}{2}$ and $m \geq n$.

Proposition 1.2. [4, 5] $I_n$ can be partitioned into subsets of type $(m+1, \ldots, m+1)$, $\underbrace{m, \ldots, m}_{k_1 \text{ terms}}$ provided that $k_1 (m+1) + k_2 m = \binom{n+1}{2}$, $k_1 > 0$ and $m+1 \geq n$.

Proposition 1.3. [2] $I_n$ can be partitioned into subsets of type $(m, m, \ldots, m, \ell)$, $\underbrace{m, \ldots, m}_{k-1 \text{ terms}}$ provided that $(k-1)m + \ell = \binom{n+1}{2}$ and $m \geq n$.

We note here that the result in Proposition 1.3 was stated differently. The authors gave a sufficient and necessary condition on $m$ and $k$ such that we can always obtain $k$ mutually disjoint subsets of $I_n$ with constant sum $m$.

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1Research supported by National Science Council of the Republic of China (NSC79-0208-M009-33).
In this paper, we show that $I_n$ can be partitioned into subsets of type $(m, m + 1, \ldots, m + k - 2, \ell)$. With this partition we can easily obtain all the three results mentioned above.

In [3], it was conjectured by Fu that for each $k$-tuple $(m_1, m_2, \ldots, m_k)$, $m_i \geq n$, $i = 1, 2, \ldots, k$, and $\Sigma_{i=1}^{k} m_i = \binom{n+1}{2}$, $I_n$ can be partitioned into subsets of type $(m_1, m_2, \ldots, m_k)$. The conjecture can be reduced to $2n - 2 \geq m_i \geq n$, $i = 1, 2, \ldots, k$, and $\Sigma_{i=1}^{k} m_i = \binom{n+1}{2}$, which was conjectured earlier by Y. Alavi et. al. in [1]. So far, this conjecture is still unsolved. With the results we have obtained, it seems that we can make the following stronger conjecture.

**Conjecture.** $I_n$ can be partitioned into subsets of type $(m_1, m_2, \ldots, m_k)$ provided that $\Sigma_{i=1}^{k} m_i = \binom{n+1}{2}$ and $m_i \geq n$ for each $i = 1, 2, \ldots, k - 1$.

### 2. The main result

For convenience, we will use an array $A = [a_{ij}]$ with $k$ columns to represent a partition of type $(s_1, s_2, \ldots, s_k)$ where the $a_{ij}$ are distinct elements of $I_n$ and $\Sigma_{i=1}^{k} a_{ij} = s_j$. As an example,

$$
\begin{bmatrix}
3 & 1 & 6 & 2 \\
* & 4 & * & 5
\end{bmatrix}
$$

is an array which represents a partition of $I_6$ into subsets of type $(3, 5, 6, 7)$.

Now we are ready to prove

**Proposition 2.1.** Let $m, \ell, k$ be integers such that $k > 0$, $m > 0$, $0 < \ell \leq \binom{n+1}{2}$ and $(k - 1)m + \ell + \binom{k-1}{2} = \binom{n+1}{2}$. Then $I_n$ can be partitioned into subsets of type $(m, m + 1, \ldots, m + k - 2, \ell)$.

Proof: The proof will be by induction. If $m + k - 2 \leq n$, then let $A_i = m + i - 1$ for $i = 1, 2, \ldots, k - 1$ and $L = I_n \setminus \bigcup_{i=1}^{k-1} A_i$, then we have the partition. If $m + k - 2 > n$ but $m \leq n$, then there exists a $j$, $1 \leq j < k - 1$, such that $m + j - 1 = n$; then let $A_i = m + i - 1$ for $i = 1, 2, \ldots, j$. Let $n' = m - 1$; then $\Sigma_{i=j+1}^{k-1} (m + i - 1) + \ell = \binom{n'+1}{2}$. By the induction hypothesis $I_{n'}$ can be partitioned into subsets of type $(m + j, m + j + 1, \ldots, m + k - 2, \ell)$. So without loss of generality, we can assume that $m > n$ and consider three cases.

**Case 1.** $m \geq 2n - 2k + 3$

Let $B_i = \{n-i+1, n-2k+i+2\}$ for $i = 1, 2, \ldots, k - 1$; then by the induction hypothesis, $I_{n-k+2}$ can be partitioned into subsets of type $(m - 2n + 2k - 3, m - 2n + 2k - 2, \ldots, m - 2n + 3k - 5, \ell)$, say $A'_1, A'_2, \ldots, A'_{k-1}, L$. Then $A'_i \cup B'_1$, $A'_2 \cup B_2, \ldots, A'_{k-1} \cup B_{k-1}, L$ are the subsets of type $(m, m+1, \ldots, m+k-2, \ell)$.

**Case 2.** $m < 2n - 2k + 3$ and $m + \lfloor \frac{k-1}{2} \rfloor - 1 \geq 2n - 2k + 3$.
In this case we can find $j$ such that $m + j - 1 = 2n - 2k + 3$ and $j \leq \left\lfloor \frac{k}{2} \right\rfloor$.

Consider the array $C_1$ in Figure 2.1.

As shown in $C_1$, if we can partition the set $I_{n-2k+2}$ into subsets of type $(m - 2n + 2k + 2j - 4, m - 2n + 2k + 2j - 3, \ldots, m - 2n + 3k - 5, \ell)$ which is represented by $C'_1$, then we are done. Since this can be obtained by the induction hypothesis, we have the proof of this case.

**Case 3.** $m + \left\lfloor \frac{k}{2} \right\rfloor - 1 < 2n - 2k + 3$.

In this case we consider two situations. If $k - 1$ is odd, then consider the following array. Let $C_0$ be the array in Figure 2.2.

$$\left[ \begin{array}{cccccc}
-n-k+3, & n-k+5, & \ldots, & n-1, & n-k+2, & n-k+4, \ldots, & n \\
m-n+k-3, m-n+k-4, \ldots, m-n+\frac{1}{2}k-1, m-n+\frac{3}{2}k-3, m-n+\frac{3}{2}k-4, \ldots, m-n+k-2 \\
\end{array} \right]$$

**Figure 2.2**

Let $A_i = \{\text{the entries in the } i\text{-th column of } C_0\}$ and $L = I_n \setminus \cup_{i=1}^{k-1} A_i$; then $A_1, A_2, \ldots, A_{k-1}, L$ are the subsets of type $(m, m+1, \ldots, m+k-2, \ell)$.

If $k - 1$ is even, then consider the array $C_e$ below.

$$\left[ \begin{array}{cccccc}
n-k+3, & n-k+5, & \ldots, & n, & n-k+2, & n-k+4, \ldots, n-1 \\
m-n+k-3, m-n+k-4, \ldots, m-n+\frac{1}{2}(k-1)-1, m-n+\frac{3}{2}(k-1)-1, m-n+\frac{3}{2}(k-1)-2, \ldots, m-n+k-1 \\
\end{array} \right]$$

**Figure 2.3**

As for the case when $k - 1$ is odd, we have the desired partition.

**Corollary 2.2.** [2,5] Let $k$, $m$, $\ell$ be positive integers such that $m \geq n$, $0 < \ell \leq \binom{n^*}{2}$ and $(k - 1)m + \ell = \binom{n+1}{2}$. Then $I_n$ can be partitioned into subsets of type $(m, m, \ldots, m, \ell)$.

**Proof:** Since $\sum_{i=1}^{k-1} (m - (n - i + 1)) + \ell = \binom{n+k^*}{2}$, by Proposition 2.1 $I_{n-k+1}$ can be partitioned into subsets of type $(m - n, m - n + 1, \ldots, m - n + k - 2, \ell)$, say, $A_1', A_2', \ldots, A_{k-1}', L$. Let $A_i = A_i' \cup \{n - i + 1\}$ for $i = 1, 2, \ldots, k - 1$; then $A_1, A_2, \ldots, A_{k-1}, L$ are the subsets of type $(m, m, \ldots, m, \ell)$. [1]
Corollary 2.3. [4,5] Let \( k_1, k_2, m \) be integers such that \( m \geq n \) and \( k_1(m+1) + k_2m = \binom{n+1}{2} \), then \( I_n \) can be partitioned into subsets of type 
\[
\{m+1, m+1, \ldots, m+1, m, m, \ldots, m\}.\]

Proof: Since \( \Sigma_{i=1}^{k_1} [(m + 1) - (n - i + 1)] + \Sigma_{i=1}^{k_2} [m - (n - k_1 - i + 1)] = \binom{n-k_1-k_2+1}{2} \), then by Proposition 2.1 \( I_{n-k_1-k_2} \) can be partitioned into subsets 
\( A'_1, A'_2, \ldots, A'_{k_1+k_2-1}, L \) of type \( \{m-n+1, m-n+2, \ldots, m-n+k_1+k_2-1, m-n+k_1\} \). Let \( A_i = A'_i \cup \{n-i+1\} \) for \( i \in \{1, 2, \ldots, k_1+k_2\}\) \( \setminus \{k_1+1\} \) and \( A_{k_1+1} = L \cup \{n-k_1\} \). Then \( A_1, A_2, \ldots, A_{k_1+k_2} \) are the subsets of type 
\[
\{m+1, m+1, \ldots, m+1, m, m, \ldots, m\}.\]

Thus, we have given a shorter proof for all the results mentioned above in Proposition 1.1, 1.2, 1.3. Moreover, we have further evidence for believing that the stronger conjecture mentioned in this paper is a true conjecture.

References

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