Two Classes of Graphs Which Have Ascending Subgraph Decompositions

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Abstract. Let $G$ be a graph with $\binom{m+1}{2}$ edges. Then we will say that $G$ has an ascending subgraph decomposition if the edge set of $G$ can be partitioned into $n$ sets generating graphs $G_1, G_2, \ldots, G_n$ such that $|E(G_i)| = i$ (for $i = 1, 2, \ldots, n$) and $G_i$ is isomorphic to a subgraph of $G_{i+1}$ for $i = 1, 2, \ldots, n-1$. In this paper, we show that a split graph with $\binom{m+1}{2}$ edges and $K_{m+3} \setminus S$, where $S$ is a subgraph of size $2n+3$, both have ascending subgraph decomposition.

Dedicated to Roger Entringer on the occasion of his 60th birthday

1. Introduction

In [1], the authors give the following partition conjecture.

Conjecture. Let $G$ be a graph on $\binom{n+1}{2}$ edges. Then the edge set of $G$ can be partitioned into $n$ sets generating graphs $G_1, G_2, \ldots, G_n$ such that $|E(G_i)| = i$ (for $i = 1, 2, \ldots, n$) and $G_i$ is isomorphic to a subgraph of $G_{i+1}$ for $i = 1, 2, \ldots, n-1$.

A graph $G$ which can be partitioned as described in the conjecture will be said to have an ascending subgraph decomposition (abbreviated ASD). The graphs $G_1, G_2, \ldots, G_n$ are members of such a decomposition. The conjecture has been verified for special classes of graphs. We list some of them as theorems.

Theorem 1.1. [2,7] Any forest with $\binom{n+1}{2}$ edges has an ASD.

Theorem 1.2. [5] If a graph $G$ has $\binom{n+1}{2}$ edges and maximum degree $\Delta(G) \leq \lfloor (n-1)/2 \rfloor$, then $G$ has an ASD with each member a matching.

Theorem 1.3. [3] If $G$ is a graph with $\binom{n+1}{2}$ edges, and $\Delta(G) < \lfloor (2 - \sqrt{2})n \rfloor$, then $G$ has an ASD. Moreover, if $G$ is a forest with $\binom{n+1}{2}$ edges, and $\Delta(G) < \lfloor (3 - \sqrt{3})n/2 \rfloor$, then $G$ has an ASD.

Theorem 1.4. [6] A complete bipartite graph with $\binom{n+1}{2}$ edges has an ASD.

A graph $G = (V, E)$ is called a split graph if its vertex set can be partitioned into two disjoint subsets $U$ and $W$ such that the graph induced by $U$ is a complete graph and the graph induced by $W$ is an empty graph.

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In this paper, we first prove that a split graph with \( \binom{n+1}{2} \) edges has an ASD, and then we show that the graph \( K_{n+3} \setminus S \) has an ASD provided that \( S \) is a subgraph of \( K_{n+3} \) with \( 2n + 3 \) edges.

2. The main result

First, we show that a split graph with \( \binom{n+1}{2} \) edges has an ASD.

**Theorem 2.1.** A split graph \( G \) with \( \binom{n+1}{2} \) edges has an ASD with each member a star.

**Proof:** The proof will be by induction on \( n \). It is trivial for small \( n \). Let the assertion be true for \( n = m \) and assume that \( G \) has size \( \binom{m+2}{2} \). Let \( V(G) = \{ u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_t \} \) where \( \{ u_1, u_2, \ldots, u_k \} \) induce a complete graph and \( \{ v_1, v_2, \ldots, v_t \} \) induce an empty graph, and \( \deg(u_1) = \Delta(G) \). First, we have \( \deg(u_1) \geq m + 1 \). For otherwise, \( \Delta(G) \leq m \) and so \( |E(G)| \leq \binom{m}{2} + k(m - k + 1) = \frac{k(k-1)}{2} + km - k = \frac{k}{2}(2m-k+1) \leq \frac{m(m+1)}{2} = \binom{m+1}{2} \). If \( \deg(u_1) = m + 1 \), delete \( u_1 \) from \( G \). Then \( G' = G \setminus u_1 \) has \( \binom{m}{2} \) edges and has an ASD with each member a star by the induction hypothesis. Hence \( G \) has an ASD with each member a star. If \( \deg(u_1) > m + 1 \), delete exactly \( m + 1 \) edges which are incident with \( u_1 \) beginning with the edges of the form \( \{ u_1, v_j \} \). There are two cases to consider. In the case that \( \deg(u_1) \geq m + k \), after deleting \( m + 1 \) edges of the form \( \{ u_1, v_j \} \), we have \( G' \) which is also a split graph with \( \binom{m+1}{2} \) edges. By induction we conclude that \( G \) has an ASD with each member a star. If \( \deg(u_1) < m + k \), let \( G' \) be the resulting graph after deleting \( m + 1 \) edges. Then \( u_1 \) is not adjacent to \( v_i \) for \( i = 1, 2, \ldots, t \). It is clear that \( \{ u_2, u_3, \ldots, u_k \} \) induces a complete graph and \( \{ v_1, v_2, \ldots, v_t, u_1 \} \) induces an empty graph. Then we have a split graph with vertex set \( \{ u_2, u_3, \ldots, u_k, v_0, v_1, \ldots, v_t, u_1 \} \). It is a split graph with \( \binom{m}{2} \) edges. Thus, \( G \) has an ASD with each member a star.

Before we prove the next theorem we need a definition. A tree \( T \) is said to be a double star if \( T \) can be obtained by joining the centers of two stars with an edge. For each \( k \geq 3 \) a double star with its centers of degree \( k - 1 \) and 2 will be denoted by \( T_k \). For convenience we let \( T_1 \) be a graph with a single edge and \( T_2 \) be a star with two edges. Clearly, \( T_1, T_2, \ldots, T_k \) is a set of ascending graphs. Now we are ready to prove the theorem which extends a result in [4]. For completeness, we state the result of [4] first.

**Theorem 2.2.** [4] If \( G \) is a graph with \( \binom{n}{2} \) edges, and at most \( n + 2 \) vertices, then \( G \) has an ASD.

**Theorem 2.3.** Let \( G \) be a graph with \( \binom{n+1}{2} \) edges and \( |V(G)| \leq n + 3 \). Then \( G \) has an ASD, \( G_1, G_2, \ldots, G_n \) with \( G_i = T_i \) for \( i = 1, 2, \ldots, n \), except when \( G \) is the union of two disjoint triangles.

**Proof:** The proof will be by induction on \( n \) and it is true for \( n \leq 4 \). Since \( |V(G)| \leq n + 3 \) and \( |E(G)| = \binom{n+1}{2} \) it follows that the maximum degree \( \Delta(G) \)
of $G$ satisfies $n - 1 \leq \Delta(G) \leq n + 2$. Thus for $n \geq 5$ we have four cases to consider according to $\Delta(G)$.

Case 1. $\Delta(G) = n - 1$: Then $G$ has $n + 3$ vertices. Let $x$ be a vertex in $V(G)$ whose degree is $n - 1$, and $N(x) = \{y|x, y \in E(G)\} = \{v_1, v_2, \ldots, v_{n-1}\}$. Let $V(G) = N(x) \cup \{x\} \cup \{a, b, c\}$. It is easy to see that there exists a vertex in $N(x)$ which is adjacent to a vertex in $\{a, b, c\}$. Without loss of generality, let the edge be $\{v_1, a\}$. Now by deleting $x$ and $\{v_1, a\}$ we have a graph $G'$, where $|V(G')| \leq n + 2$ and $|E(G')| = \binom{n}{2}$. Thus by the induction hypothesis $G'$ has an ASD, $T_1, T_2, \ldots, T_{n-1}$, and an ASD of $G$ is easy to obtain.

Case 2. $\Delta(G) = n$: Let $x$ be a vertex of maximum degree in $G$ and consider the graph $G' = G \setminus x$. By the induction hypothesis, $G'$ has an ASD $G'_i \cong T_i$, $i = 1, 2, \ldots, n - 1$. Since $|V(G'_{n-1})| = n$ and $|N(x)| = n$, $|V(G'_{n-1}) \cap N(x)| \geq n + n - (n + 2) = n - 2 \geq 3$. Let $G'_{n-1}$ be as in Figure 2.1.

![Figure 2.1](image)

If $\{b, c\} \subset N(x)$, the ASD will be obtained by letting $G_n = (S_x \setminus \{b, c\}) \setminus \{x, b\}$, $G_{n-1} = (G'_{n-1} \cup \{x, b\}) \setminus \{b, c\}$, $G_{n-2} = G'_{n-2}$, $\ldots$, $G_1 = G'_1$, where $S_x$ is the star induced by the edges incident to $x$ in $G$. Otherwise we have two situations, to consider. First, only one of $b$ and $c$ is in $N(x)$. Then there exists a $u_i, 1 \leq i \leq n - 3$, such that $u_i \in N(x)$. Hence the ASD can be obtained by letting $G_n = (S_x \cup \{b, c\}) \setminus \{x, u_i\}$, $G_{n-1} = (G'_{n-1} \cup \{x, u_i\}) \setminus \{b, c\}$, and $G_{n-2} = G'_{n-2}$, $\ldots$, $G_1 = G'_1$. Secondly, neither $b$ nor $c$ is in $N(x)$, then $\{a, u_1, u_2, \ldots, u_{n-3}\} \subset N(x)$. The ASD of $G$ will be $G_1 = G'_1$, $G_2 = G'_2$, $\ldots$, $G_{n-2} = G'_{n-2}$, $G_{n-1} = (G'_{n-1} \cup \{x, a\}) \setminus \{a, u_1\}$ and $G_n = (S_x \cup \{a, u_1\}) \setminus \{x, a\}$. This concludes the proof of this case.

Case 3. $\Delta(G) = n + 1$: Let $x$ be a vertex of maximum degree in $G$ and let $N(x) = \{v_1, v_2, \ldots, v_{n+1}\}$. Then there are vertices $v_i$ and $v_j$, $1 \leq i < j \leq n+1$, such that $\{v_i, v_j\}$ is not an edge of $G$. Let $G' = (G \setminus x) \cup \{v_i, v_j\}$. By the induction hypothesis, $G'$ has an ASD $G'_i \cong T_i$, $i = 1, 2, \ldots, n - 1$. Now if $\{v_i, v_j\} \subset G'_1$ then $G'_{2} \cup S_x$ must be one of the graphs in Figure 2.2. Each of these graphs can be partitioned into $T_1, T_2, T_n$. Let $G_t = G'_t$ where $i = 3, 4, \ldots, n - 1$, $G_n = T_n$, $G_2 = T_2$ and $G_1 = T_1$. Then we have the ASD for $G$. If $\{v_i, v_j\} \subset G'_2$ then $(G'_2 \cup S_x) \setminus \{v_i, v_j\}$ must be one of the graphs of Figure 2.3 and by a similar argument, $G$ has an ASD with $G'_i \cong T_i$ for $i = 1, 2, \ldots, n$.

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Suppose, then, that \( \{v_i, v_j\} \in G^{\prime}_z \) where \( 3 \leq \ell \leq n - 1 \). We consider three cases.

(i) \( \{v_i, v_j\} = \{a, b\} \). Let \( G_n = (S_x \cup \{b, c\}) \setminus \{(a, x) \cup \{x, y\}\) and \( G_\ell = (G^{\prime}_z \cup \{a, x\} \cup \{x, y\}) \setminus \{(b, c) \cup \{a, b\}\) where

\[
y \in \begin{cases}
\{c\} & \text{if } \{b, c\} \subset N(x); \\
N(x) \setminus V(G^{\prime}_z) & \text{otherwise}.
\end{cases}
\]

and \( G_i = G^{\prime}_i \) where \( i \in \{1, 2, \ldots, n - 1\} \setminus \{\ell\} \).

(ii) \( \{v_i, v_j\} = \{b, c\} \), let \( G^{\prime}_i = \{w, z\} \) and W.L.O.G. let \( w \in N(x) \). If \( a \in N(x) \) then let \( G_n = (S_x \cup \{a, b\}) \setminus \{(a, x) \cup \{x, c\}\) and \( G_\ell = G^{\prime}_z \cup \{(a, x) \cup \{x, c\}\) and \( G_i = G^{\prime}_i \) for \( i \in \{1, 2, \ldots, n - 1\} \setminus \{\ell\} \). If \( a \notin N(x) \) then let \( G_\ell = (G^{\prime}_z \cup \{y, x\}) \setminus \{b, c\} \) where \( y \in \{b, u_1\} \setminus \{w\} \).

Then \( (S_x \cup G^{\prime}_i) \setminus \{x, y\} \) must be (a) or (b) in Figure 2.5 which clearly can be decomposed into \( G_n \cong T_n \) and \( G_1 \cong T_1 \) and \( G_i = G^{\prime}_i \) where \( i \in \{2, 3, \ldots, n - 1\} \setminus \{\ell\} \).

(iii) \( \{v_i, v_j\} = \{a, u_p\} \) where \( 1 \leq p \leq \ell - 2 \). Since \( |N(x)| = n + 1 \), at least one of \( b \) or \( c \) is in \( N(x) \). As in (i) we can decompose \( (S_x \cup G^{\prime}_n) \setminus \{a, u_p\} \) into \( G_n = T_n \) and \( G_\ell = T_\ell \). Let \( G_i = G^{\prime}_i \) for \( i \in \{1, 2, \ldots, n - 1\} \setminus \{\ell\} \).

This gives an ASD for \( G \).

Case 4. \( \Delta(G) = n+2 \): If there exists a vertex \( x \) such that \( n-1 \leq \deg(x) \leq n+1 \) then go back to Case 1, 2 or 3. Assume, then, that \( G \) contains \( r(\geq 1) \) vertices.

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which have degree \( n+2 \), a vertex \( z \) of degree \( n-k \) \((k \geq 2)\) and no vertex of degree \( p \) such that \( n-k < p < n+2 \). Then \((n-k)(n+3-r) + r(n+2) \geq n^2 + n\). It is easy to check that \( r \geq k \geq 2 \) and \( n-k \geq r \). Since at least one neighbor of \( z \) has degree \( n+2 \), we can delete all edges incident to \( z \) and one another edge \( e \) to form \( T_{n-k+1} \). We claim that \( \tilde{G} = (G \setminus z) \setminus e \) can be decomposed into \( T_1, T_2, \ldots, T_{n-k}, T_{n-k+2}, T_{n-k+3}, \ldots, T_n \), i.e., a graph of order \( n+2 \) with \( \binom{n+1}{2} - (n-k+1) \) edges which has at least \( k-1 \) vertices with degree \( n+1 \) can be decomposed into \( T_1, T_2, \ldots, T_{n-k}, T_{n-k+2}, \ldots, T_n \) whenever \( n-k \geq 2 \).

Proof of the claim: We prove the claim by induction and it is true for \( n=4 \) (since \( n-k \geq 2 \) we have that \( n \geq 4 \)). Assume \( \tilde{G} \) has \( n+2 \) vertices and at least \( k-1 \) vertices which have degree \( n+1 \). Let \( x \) be a vertex of degree \( n+1 \) and \( N(x) = \{v_1, v_2, \ldots, v_{n+1}\} \). Then there are vertices \( v_i \) and \( v_j \), \( 1 \leq i < j \leq n+1 \), such that \( \{v_i, v_j\} \) is not an edge of \( \tilde{G} \). Let \( \tilde{G}' = (\tilde{G} \setminus x) \cup \{v_i, v_j\} \) which by induction can be decomposed into \( \tilde{G}'_1, \tilde{G}'_2, \ldots, \tilde{G}'_{n-k}, \tilde{G}'_{n-k+2}, \ldots, \tilde{G}'_{n-1} \) if \( n > n-k+2 \), or \( \tilde{G}'_1, \tilde{G}'_2, \ldots, \tilde{G}'_{n-k} \) if \( n = n-k+2 \). Let \( \{v_i, v_j\} \) be in \( \tilde{G}'_\ell \). Then by the same argument as in Case 3 \( G'' \cup S_x \), can be decomposed into \( T_1, T_2, \ldots, T_{n-k}, T_{n-k+2}, \ldots, T_n \).

Thus we have the proof.

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