On Latin \((nxnx(n-2))\)-Parallelepipeds

HUNG-LIN FU
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1. Introduction

Let \(A_1, A_2, \ldots, A_k\) be pairwise disjoint latin squares of (the same) \(n\) elements. The ordered \(k\)-tuple \(A = (A_1, A_2, \ldots, A_k)\) is called a latin \((n \times n \times k)\)-parallelepiped. In the case \(k = n\), \(A\) is called a latin cube of order \(n\).

In [1], P. Horák shows that, for all \(n = 2^k\), \(k \geq 3\), there exists a latin \((n \times n \times (n-2))\)-parallelepiped that cannot be embedded in any latin cube of order \(n\).

In this paper, generalizing the idea of [1], we prove that for each \(n \geq 12\) there exists a latin \((n \times n \times (n-2))\)-parallelepiped that cannot be embedded in any latin cube of order \(n\). Moreover, we show that a latin \((n \times n \times (n-2))\)-parallelepiped can always be embedded in a latin cube of order \(m\) with \(m \geq 4n\).

2. Main Theorems

We start with the construction of a latin \((6 \times 6 \times 4)\)-parallelepiped which cannot be embedded in a latin cube of order 6. (Figure 2.1).

\[
A_1 = \begin{bmatrix}
1 & 2 & 4 & 3 & 5 & 6 \\
2 & 1 & 3 & 5 & 6 & 4 \\
4 & 3 & 2 & 6 & 1 & 5 \\
3 & 5 & 6 & 1 & 4 & 2 \\
5 & 6 & 1 & 4 & 2 & 3 \\
6 & 4 & 5 & 2 & 3 & 1
\end{bmatrix} \quad A_2 = \begin{bmatrix}
2 & 1 & 5 & 4 & 6 & 3 \\
1 & 3 & 2 & 6 & 4 & 5 \\
6 & 2 & 3 & 1 & 5 & 4 \\
5 & 6 & 4 & 2 & 3 & 1 \\
3 & 4 & 6 & 5 & 1 & 2 \\
4 & 5 & 1 & 3 & 2 & 6
\end{bmatrix}
\]
LEMMA 2.1. \((A_1, A_2, A_3, A_4)\) is a latin \((6 \times 6 \times 4)\)-parallelepiped which cannot be embedded in a latin cube of order 6.

PROOF. Let \(C = [S_{i,j}]\) be the 6x6 array where \(S_{i,j} = \{1, 2, \ldots, 6\} \setminus \{A_1(i, j), A_2(i, j), A_3(i, j), A_4(i, j)\}\), \((A_k(i, j))\) is the \((i, j)\)-entry of the latin square \(A_k\). (Figure 2.2.) In order to embed \((A_1, A_2, A_3, A_4)\) into a latin cube of order 6, we have to find \(A_5\) and \(A_6\) such that \(A_1, A_2, \ldots, A_6\) are pairwise disjoint latin squares. Hence, if \(\{a, b\} = S_{i,j}\) \(\{c, d\} = S_{i',j'}\) and \(i = i'\) or \(j = j'\) (not both), then \(A_k(i, j) = a\) and \(A_k(i', j') = c\) should imply that \(b \neq d\) (\(k = 5\) or 6). We start with the entry \(A_5(1, 2)\). We will use \(\rightarrow A_5(i, j)\) to denote the next entry to be picked. (1) \(A_5(1, 2) = 3 \rightarrow A_5(2, 2) = 4 \rightarrow A_5(3, 2) = 6 \rightarrow A_5(5, 2) = 5 \rightarrow A_5(6, 2) = 2\), but \(A_5(1, 2) = 3 \rightarrow A_5(1, 1) = 4 \rightarrow A_5(1, 3) = 6 \rightarrow A_5(6, 3) = 2\), which is not possible for \(A_5\). Similarly, (2) \(A_5(1, 2) = 4 \rightarrow A_5(4, 2) = 3 \rightarrow A_5(6, 2) = 1 \rightarrow A_5(5, 2) = 2 \rightarrow A_5(3, 2) = 5 \rightarrow A_5(3, 5) = 6 \rightarrow A_5(3, 6) = 2 \rightarrow A_5(3, 3) = 1 \rightarrow A_5(2, 3) = 5 \rightarrow A_5(4, 3) = 3\) which contradicts \(A_5(4, 2) = 3\). Since it is not possible to find \(A_5\), we have the proof.

\[
\begin{array}{cccccc}
4,6 & 3,4 & 2,6 & 2,5 & 1,3 & 1,5 \\
4,6 & 4,6 & 1,5 & 2,3 & 1,5 & 2,3 \\
3,5 & 5,6 & 1,4 & 3,4 & 2,6 & 1,2 \\
1,2 & 1,3 & 3,5 & 4,6 & 2,5 & 4,6 \\
1,2 & 2,5 & 3,4 & 1,6 & 3,4 & 5,6 \\
3,5 & 1,2 & 2,6 & 1,5 & 4,6 & 3,4
\end{array}
\]

Figure 2.2

LEMMA 2.2. A latin cube of order \(n\) can be embedded in a latin cube of order \(m\) for every \(m \geq 2n\).

PROOF. (We note that this lemma has been proved in [2]. We recall it here, since it is in preprint.) Let \(m \geq 2n\). It is well known that a latin square \(A\) of order
n can be embedded in a latin square \( L = [k_{i,j}] \) of order \( m \geq 2n \). Set \( L_t = L \)
and construct \( L_t, t = 2, 3, \ldots, m \), by letting the \((i, j)\)-entry in \( L_t \) be 
\[ \varphi_{1,1} \varphi_{1,2} \cdots \varphi_{1,m} \]
where \( \alpha_t \) is the permutation 
\[ \varphi_{t,1} \varphi_{t,2} \cdots \varphi_{t,m} \]. It is easy to see that \( (L_1, \)
\( L_2, \ldots, L_m) \) is a latin cube of order \( m \) and contains a latin sub-cube of order \( n \)
generated from \( A \) in the upper-left corners of \( (L_1, L_2, \ldots, L_n) \). We can replace
this sub-cube with the original given latin cube of order \( n \). This completes the
proof.

Now we are ready for the main theorem.

THEOREM 2.3. For each \( n \geq 12 \), there exists a latin \((nxnx(n-2))\)-
parallelepiped which cannot be embedded in a latin cube of order \( n \).

PROOF. Let \( n \) be any positive integer \( \geq 12 \). By Lemma 2.2, there exists a
latin cube \( L = (L_1, L_2, \ldots, L_n) \) of order \( n \), which contains a latin cube \( B = (B_1, B_2, \ldots, B_6) \) of order \( 6 \), in the upper-left corners of \( (L_1, L_2, \ldots, L_n) \). By
replacing \( B_i \) with \( A_i \) of Figure 2.1, \( i = 1, 2, 3, 4 \), it is easy to see \( (L_1, L_2, L_3, L_4, L_7, L_8, \ldots, L_n) \) is a latin \((nxnx(n-2))\)-parallelepiped which cannot be
embedded in a latin cube of order \( n \).

LEMMA 2.4. Any latin \((nxnx(n-2))\)-parallelepiped can be embedded in a
latin cube of order \( 2n \).

PROOF. Let \( A = (A_1, A_2, \ldots, A_{n-2}) \) be a latin \((nxnx(n-2))\)-parallelepiped
and \( C \) be the nxnx array \( [S_{i,j}] \) such that \( S_{i,j} = \{ 1, 2, \ldots, n \} \setminus \{ A_k(i,j) \mid k = 1, 2, \ldots, n-2 \} \). It is not difficult to see \( \{ S_{i,1}, S_{i,2}, \ldots, S_{i,n} \} \) satisfies the Hall's
condition. Hence we can construct an nxnx array \( B_{n-1} = [B_{n-1}(i,j)] \) such that
the \( i \)-th row is an SDR (system of distinct representatives) of \( \{ S_{i,1}, S_{i,2}, \ldots, S_{i,n} \} \). It is a direct result that each row of the nxnx array \( B_n = [B_n(i,j)] \) has
distinct elements where \( B_n(i,j) = S_{i,j} \setminus \{ B_{n-1}(i,j) \} \). Now let \( B'_{n-1} = [B'_{n-1}(i,j)] \)
and \( B'_{n-1}(i,j) = B_{n-1}(i,j) + n \) if there exists \( i' > i \) such that
\( B_{n-1}(i',j) = B_{n-1}(i,j) \), otherwise \( B'_{n-1}(i,j) = B_{n-1}(i,j) \). Similarly we construct
\( B'_n \). Moreover, we let \( A_{n-1} = B'_{n-1} \), and \( A_n(i,j) = B'_n(i,j) + n \) if \( B'_n(i,j) \) occurs
in the \( j \)-th column of \( A_{n-1} = B'_{n-1} \), otherwise \( A_n(i,j) = B'_n(i,j) \). We are ready
to construct a latin cube of order \( 2n \) which contains the parallelepiped \( (A_1, A_2,
\ldots, A_{n-2}) \). (Figure 2.3.) In the figure, we define \( C_k = [C_k(i,j)] \) as follows:
\( C_k(i,j) = A_k(i,j) + n \), if \( A_k(i,j) \leq n \); \( C_k(i,j) = A_k(i,j) - n \), if \( A_k(i,j) > n \).
It is not difficult to see that $L_k$ is a latin square for each $1 \leq k \leq n-2$, and $n+3 \leq k \leq 2n$. If $L_{n-1}$ is a latin square, so are $L_n$, $L_{n+1}$, and $L_{n+2}$. It suffices to check that $L_{n-1}$ is a latin square of order $2n$. Since, by construction, $L_{n-1}(i, j) \neq L_{n-1}(i', j)$, and $L_{n-1}(i, j) \neq L_{n-1}(i, j')$ for each $i' \neq i$ and $j' \neq j$, we conclude that $L_{n-1}$ is a latin square. $L_k(i, j) \neq L_k(i, j)$ for each $k' \neq k$ is a direct result of the way we defined $L_k$. Hence $L = (L_1, L_2, \ldots, L_{2n})$ is a latin cube of order $2n$ which contains the parallelepiped $(A_1, A_2, \ldots, A_{n-2})$. This completes the proof.

THEOREM 2.5. Any latin $(n \times n \times (n-2))$-parallelepiped can be embedded in a latin cube of order $m$ for every $m \geq 4n$.

PROOF. By Lemmas 2.2, and 2.4.
References


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