Total Chromatic Number and Chromatic Index of Split Graphs

Bor-Liang Chen and Hung-Lin Fu*
Department of Applied Mathematics
National Chiao-Tung University
Hsin-Chu, Taiwan, R.O.C.

M.T. Ko
Institute of Information Science
Academia Sinica
Taipei, Taiwan

ABSTRACT. A graph $G$ is called a split graph if its vertex set can be partitioned into two subsets $U$ and $V$ such that $U$ induces a clique and $V$ is independent in $G$. In this paper, we study the total chromatic number and chromatic index of split graphs. Some results are obtained in the direction of solving the entire problem.

1. Introduction

Throughout this paper, all graphs are simple, finite, and undirected. Let $G$ be a graph and let its complementary graph, vertex set, edge set, maximum degree, order, and size be denoted by $\overline{G}$, $V(G)$, $E(G)$, $\Delta(G)$, $|V(G)|$, and $|E(G)|$, respectively. A graph $G$ is called a split graph if its vertex set can be partitioned into two subsets $U$ and $V$ such that $U$ induces a clique and $V$ is independent in $G$. Let the complete graph of order $n$ and the null graph of order $r$ be denoted by $K_n$ and $O_r$, respectively. The join of $K_n$ and $O_r$, $K_n + O_r = S(n,r)$, is a complete split graph.

An edge coloring $\sigma$ of a graph $G$ is a mapping $\sigma : E(G) \rightarrow \{1, 2, \ldots \}$ such that two adjacent edges have distinct images. The chromatic index of $G$, $\chi'(G)$, is the smallest integer $k$ such that $G$ has an edge coloring having

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image set \( \{1, 2, \ldots, k\} \). The well-known Vizing’s theorem says that \( \chi'(G) \) is either \( \Delta(G) \) or \( \Delta(G) + 1 \). A graph \( G \) is said to be class one (resp. class two) if \( \chi'(G) = \Delta(G) \) (resp. \( \Delta(G) + 1 \)).

A total coloring \( \pi \) of a graph \( G \) is a mapping \( \pi : V(G) \cup E(G) \to \{1, 2, \ldots\} \) such that (1) no two adjacent vertices or edges have the same image; (2) the image of each vertex is distinct from the images of its incident edges. The total chromatic number \( \chi_T(G) \) of a graph \( G \) is the smallest integer \( k \) such that \( G \) has a total coloring having image set \( \{1, 2, \ldots, k\} \). Clearly, \( \chi_T(G) \geq \Delta(G) + 1 \). Behzad [1] and Vizing [10] independently made the following conjecture:

**Total Coloring Conjecture (TCC):** For any graph \( G \), \( \chi_T(G) \leq \Delta(G) + 2 \).

This conjecture was proved for several classes of graphs [6, 7, 12, 13, 15]. Now if the TCC holds for a certain class of graphs, then we shall say that \( G \) is type one if \( \chi_T(G) = \Delta(G) + 1 \) and is type two if \( \chi_T(G) = \Delta(G) + 2 \).

In this paper, we mainly prove that the split graphs satisfy the TCC and that a split graph \( G \) is type one if \( \Delta(G) \) is even. We also prove that a split graph \( G \) is class one if \( \Delta(G) \) is odd.

2. The Complete Split Graph

It is easier to determine the total chromatic number and chromatic index of complete split graphs, \( S(n, r) \), than to determine those of general split graphs. First, we need several lemmas.

**Lemma 2.1.** For any subgraph \( H \) of a graph \( G \), \( \chi'(H) \leq \chi'(G) \) and \( \chi_T(H) \leq \chi_T(G) \).

**Lemma 2.2.** ([2]) \( K_n \) is class one and type two if and only if \( n \) is even.

**Lemma 2.3.** ([9]) If \( G \) is a graph of order \( 2n + 1 \) and \( \Delta(G) = 2n \), then \( G \) is class one if and only if \( |E(G)| \geq n \).

**Lemma 2.4.** ([5]) Let \( G \) be a graph of order \( 2n \) and \( \Delta(G) = 2n - 1 \). Then \( G \) is type one if and only if \( \alpha'(G) + |E(G)| \geq n \), where \( \alpha'(G) \) is the edge independence number of \( G \).

Now we are ready to prove two theorems on determining the chromatic index and total coloring number of \( S(n, r) \).

**Theorem 2.5.** \( S(n, r) \) is class two if and only if \( n + r \) is odd and \( \binom{r}{2} < \frac{n + r}{2} \).

**Proof:** Note that \( \Delta(S(n, r)) = n + r - 1 \) and \( S(n, r) \) is a subgraph of \( K_{n+r} \). If \( n + r \) is even, by Lemma 2.1 and Lemma 2.2, \( \chi'(S(n, r)) \leq \chi'(K_{n+r}) = n + r - 1 = \Delta(S(n, r)) \). Hence, \( S(n, r) \) is class one as \( n + r \) is
even. If \( n + r \) is odd, then by Lemma 2.3, \( S(n, r) \) is class one if and only if \\
\(|E(S(n, r))| \geq (n + r - 1)/2, \) i.e. \( \binom{n}{2} \geq (n + r - 1)/2. \) Thus, we conclude \( S(n, r) \) is class two if and only if \( r + n \) is odd and \( \binom{n}{2} < \left\lfloor \frac{n+1}{2} \right\rfloor. \) \( \square \)

We note here that a complete split graph \( S(n, r) \) can be considered as a complete \((r + 1)\)-partite graph and the chromatic index of a complete partite graph was determined by Hoffman and Rodger in [7], which shows a complete partite graph is class two if and only if it is overall.

**Theorem 2.6.** \( S(n, r) \) is type two if and only if \( n + r \) is even and \( \binom{n}{2} < \left\lfloor \frac{n}{2} \right\rfloor. \)

**Proof:** If \( n + r \) is odd, then by Lemma 2.2, \( K_{n+r} \) is type one. Since \\
\( \Delta(S(n, r)) = n + r - 1 = \Delta(K_{n+r}) \), by Lemma 2.1, \( S(n, r) \) is also type one, as \( n + r \) is odd. If \( n + r \) is even, by Lemma 2.4, \( S(n, r) \) is type one if and only if \\
\( \left\lfloor \frac{n}{2} \right\rfloor + \binom{n}{2} \geq (n + r)/2; \) i.e., \( S(n, r) \) is type two if and only if \\
\( \binom{n}{2} < \left\lfloor \frac{n}{2} \right\rfloor. \) \( \square \)

For the total chromatic number of the complete partite graph, only partial results has been obtained [3, 4, 11].

3. The Split Graphs

Let \( S \subseteq V(G) \) and let \( G[S] \) denote the subgraph of \( G \) induced by \( S \). Hence the vertex-set of a split graph \( G \) can be expressed as a disjoint union of two sets \( U \) and \( V \) such that \( G[U] \cong K_n \) and \( G[V] \cong O_r \), where \( |U| = n \) and \( |V| = r \). In what follows, we assume our split graph is of this form.

First, we show that the split graphs satisfy the TCC.

**Lemma 3.1.** ([14]) Let \( G \) be a graph of order \( v \) having \( \Delta(G) = v - k \). If \( G \) contains an induced subgraph \( O_{k-1} \), then \( \chi_T(G) \leq \Delta(G) + 2 \), i.e., \( G \) satisfies the TCC.

**Theorem 3.2.** The split graphs satisfy the TCC.

**Proof:** Let \( G \) be a split graph. Then \( \Delta(G) = n + r - k \), where \( 1 \leq k \leq r+1 \). By the fact that \( G \) contains an induced subgraph \( O_{(r+1)-1} = O_r \) and Lemma 3.1, we conclude the proof. \( \square \)

Next we present two results on concerning split graphs

**Lemma 3.3.** Let \( G \) be a split graph. If \( \Delta(G) \geq \max\{\text{deg}(v) \mid v \in V\} + n \), then \( G \) is type one.

**Proof:** It is not difficult to see that \( \Delta(G) = \max\{\text{deg}(u) \mid u \in U\} \). Consider the bipartite subgraph \( H = G \setminus E(G[U]) \). By assumption, we have \( \text{deg}(v) \leq \Delta(G) - n \) for all \( v \in V \) and \( \text{deg}_H(u) = \Delta(G) - n + 1 \) for some \( u \in U, \Delta(H) = \Delta(G) - n + 1 \).

Now the proof follows by giving a total coloring \( \pi \) directly. First, if \( n \) is odd, then \( G[U] \cong K_n \) is type one. Let \( \pi_1 \) be a total coloring of \( G[U] \)
which uses colors 1, 2, ..., n. Since $G \setminus E(G[U])$ is a bipartite graph with maximum degree $\Delta(G) - n + 1$, by the well-known König's theorem, there exists an edge-coloring $\pi_2$ of $G \setminus E(G[U])$ which uses $\Delta(G) - n + 1$ colors $n + 1, n + 2, \ldots, \Delta(G) + 1$. Now we can define a total coloring for $G, \pi$, by letting $\pi(v) = \pi_1(v)$ if $v \in U$, $\pi(e) = \pi_1(e)$ if $e \in E(G[U])$, $\pi(h) = \pi_2(h)$ if $h \in E(G) \setminus E(G[U])$ and $\pi(v)$ is a color in $\{n + 1, n + 2, \ldots, \Delta(G) + 1\}$ missing in the edges incident to $v$, for $v \in V$. Thus $G$ is type one.

Second, consider the case where $n$ is even. Let $\pi_1$ be a total coloring of $G[U]$ with colors 1, 2, ..., $n + 1$, and $\pi_2$ be an edge-coloring of $G \setminus E(G[U])$ with colors $n + 2, \ldots, \Delta(G) + 2$. Let $\pi$ be a coloring of $G$ such that (i) for $v \in U$, $\pi(v) = \pi_1(v)$, (ii) for $e \in E(G[U])$, $\pi(e) = \pi_1(e)$, (iii) for $h \in E(G) \setminus E(G[U])$ and $\pi_2(h) \neq \Delta(G) + 2$, $\pi(h) = \pi_2(h)$, (iv) for $g = uv, u \in U, v \in V$ and $\pi_2(g) = \Delta(G) + 2$, $\pi(g)$ is the missing color of $\{1, 2, \ldots, n + 1\}$ in the star with center $u$, (v) for $v \in V$ incident to an edge colored $\Delta(G) + 2$ in the edge coloring $\pi_2$, $\pi(v)$ is a missing color of $\{n + 2, n + 3, \ldots, \Delta(G) + 1\}$ in the edges incident to $v$, and (vi) for $v \in V$ incident to no edge colored $\Delta(G) + 2$, $\pi(v)$ is the color in $\{1, 2, \ldots, n + 1\} \setminus \{\pi_1(u) : u \in U\}$. Thus we obtain a type one coloring $\pi$ for $G$, and we have the proof.

For the edge-coloring, we have a similar result. Since the proof is similar, we omit it.

**Lemma 3.4.** Let $G$ be a split graph. If $\Delta(G) \geq \max\{\deg(v) \mid v \in V\} + n$, then $G$ is class one.

Before proving the main theorems, we need some definitions. A color diagram $C = (R_1, \ldots, R_k)$ of frame $d = (d_1, d_2, \ldots, d_k)$ is an ordered set of color arrays, where color array $R_i = [c_{i,1}, \ldots, c_{i,d_i}]$, of length $d_i$, consists of distinct colors for all $1 \leq i \leq k$. Color diagram $C$ is called monotonic if the color $c_{i,j}$ occurs at most $d_i - j$ times in $R_1, \ldots, R_{i-1}$ for all $1 \leq i \leq k$, $1 \leq j \leq d_i$.

Let $B = (U, V, E)$ be a bipartite graph with bipartition $(U, V)$ and edge set $E$. An edge coloring scheme with respect to $U$ is a collection of sets of colors $S = \{S_u\}_{u \in U}$, where $S_u$ is a set of $\deg_B(u)$ distinct colors for $u \in U$. Scheme $S$ is fulfilled by an edge coloring of $B$ if the set of colors occurring on edges incident to $u$ is exactly $S_u$ for all $u \in U$. An edge coloring scheme $S$ can be arranged into a color diagram by assigning an order to colors in $S_u$, for all $u \in U$ and an order to $\{S_u\}_{u \in U}$.

**Lemma 3.5.** If $S$ can be arranged into a monotonic color diagram, then $B$ has an edge coloring which fulfills $S$.

**Proof:** Let $C = (R_{u_1}, \ldots, R_{u_k})$ be a monotonic color diagram arranged from $S$, where $k = |U|$ and the set of colors in $R_{u_i}$ is exactly $S_{u_i}$. We are going to give an edge coloring to prove the lemma. We color the edges incident to $u_1$ first, then those incident to $u_2, \ldots$, and so on. For each $i$, we
find an edge incident to \( u_i \) that we can color with \( c_{i,1} \) first, and then an edge we can color with \( c_{i,2}, \ldots \), until we reach the last edge, which is colored with \( c_{i,\deg(u_i)} \). By the time we are looking for an edge to color \( c_{i,j} \), the color \( c_{i,j} \) has occurred at most \( \deg(u_i) - j \) times by the monotonicity of \( C \). Since there are \( \deg(u_i) - j + 1 \) edges incident to \( u_i \) not colored yet, there exists at least one edge incident to \( u_i \) which can be colored with \( c_{i,j} \). Hence, we can color all the edges incident to \( u_i \) by using the colors \( c_{i,1}, \ldots, c_{i,\deg(u_i)} \). Notice that the set of colors that occur on the edges incident to \( u_i \) is exactly \( S_{u_i} \). Therefore, we obtain an edge coloring which fulfills scheme \( S \).

An array \([a_1, \ldots, a_s]\), which can be null, is a suffix (resp. prefix) of an array \( R = [r_1, \ldots, r_k] \) if \( a_i = r_{k-s+i} \) (resp. \( a_i = r_i \)) for all \( 1 \leq i \leq s \). A color diagram \( A = (A_1, \ldots, A_k) \) is a suffix (resp. prefix) of \( C = (R_1, \ldots, R_k) \) if \( A_i \) is a suffix (resp. prefix) of \( R_i \) for all \( 1 \leq i \leq k \). Denote the length of an array \( R \) by \( |R| \). A prefix, \( A \), of \( C \) is called increasing if \( |A_1| \leq |A_2| \leq \cdots \leq |A_k| \).

**Lemma 3.6.** If a color diagram \( C \) is monotonic, then any suffix of \( C \) is monotonic.

**Proof:** The lemma is a direct result of the above definition.

A *latin square* of order \( k \) is a \( k \times k \) array based on the elements \( 1, 2, \ldots, k \) such that each element occurs in each row and each column exactly once. A latin square \( L = [l_{ij}] \) is said to be *commutative* if \( l_{ij} = l_{ji} \), for \( 1 \leq i \leq j \leq k \). \( L \) is *idempotent* provided that \( l_{ii} = i \) for each \( i = 1, 2, \ldots, k \). It is well known that an idempotent commutative latin square (abbreviated ICLS) of order \( k \) exists if and only if \( k \) is odd. Let \( M(k) = [m_{i,j}] \), where \( m_{i,j} \equiv (i + j)k \mod 2k - 1 \), \( 1 \leq m_{i,j} \leq 2k - 1 \). Then it is easy to verify that \( M(k) \) is an ICLS of order \( 2k - 1 \). Since we will use \( M(k) \) in the proof of our main theorems, we present an example for \( k = 7 \) in Figure 1.

In the following, let \( C(n, k) = (R_1, \ldots, R_n) \) denote the color diagram with \( R_i = [m_{i,n+1}, \ldots, m_{i,2k-1}], 1 \leq i \leq n \), where the \( m_{i,j} \)'s are entries of \( M(k) \) and \( n < 2k - 1 \). Denote the reverse of \( R_i \), \( [m_{i,2k-1}, \ldots, m_{i,n+1}] \), by \( R'_i \) for \( 1 \leq i \leq n \). Let \( C'(n, k) \) denote the color diagram \([R'_n, R'_{n-1}, R'_1]\). By proper color substitution, we can see that \( C(n, k) \) is isomorphic to \( C'(n, k) \). Let \( D(n, k) = (D_1, \ldots, D_n) \) be the color diagram in which \( D_i = R_i \) for each \( 1 \leq i \leq n/2 \) and \( D_i \) is the concatenation of \( m_{i,i} \) and \( R_i \) for each \( n/2 < i \leq n \), where \( R_i \) in \( C(n, k) \). Let \( D'(n, k) = (D'_1, \ldots, D'_1) \) be the color diagram in which \( D'_i = R'_i \) for each \( n/2 \leq i \leq n \) and \( D'_i \) is the concatenation of \( m_{i,i} \) and \( R'_i \) for each \( 1 \leq i < n/2 \).

By the fact that \( m_{i,j} = m_{i-1,j+1} \), for \( 1 < i \leq 2k - 1 \) and \( 1 \leq j < 2k - 1 \), it is easy to obtain the following lemmas, which will be used in the main theorem.

**Lemma 3.7.** The color diagrams \( C(n, k) \) and \( C'(n, k) \) are monotonic for all \( 1 \leq n < 2k - 1 \).
\[
\begin{array}{cccccccccccc}
1 & 8 & 2 & 9 & 3 & 10 & 4 & 11 & 5 & 12 & 6 & 13 & 7 \\
8 & 2 & 9 & 3 & 10 & 4 & 11 & 5 & 12 & 6 & 13 & 7 & 1 \\
2 & 9 & 3 & 10 & 4 & 11 & 5 & 12 & 6 & 13 & 7 & 1 & 8 \\
9 & 3 & 10 & 4 & 11 & 5 & 12 & 6 & 13 & 7 & 1 & 8 & 2 \\
3 & 10 & 4 & 11 & 5 & 12 & 6 & 13 & 7 & 1 & 8 & 2 & 9 \\
10 & 4 & 11 & 5 & 12 & 6 & 13 & 7 & 1 & 8 & 2 & 9 & 3 \\
4 & 11 & 5 & 12 & 6 & 13 & 7 & 1 & 8 & 2 & 9 & 3 & 10 \\
11 & 5 & 12 & 6 & 13 & 7 & 1 & 8 & 2 & 9 & 3 & 10 & 4 \\
5 & 12 & 6 & 13 & 7 & 1 & 8 & 2 & 9 & 3 & 10 & 4 & 11 \\
12 & 6 & 13 & 7 & 1 & 8 & 2 & 9 & 3 & 10 & 4 & 11 & 5 \\
6 & 13 & 7 & 1 & 8 & 2 & 9 & 3 & 10 & 4 & 11 & 5 & 12 \\
13 & 7 & 1 & 8 & 2 & 9 & 3 & 10 & 4 & 11 & 5 & 12 & 6 \\
7 & 1 & 8 & 2 & 9 & 3 & 10 & 4 & 11 & 5 & 12 & 6 & 13 \\
\end{array}
\]

Figure 1: An idempotent commutative Latin square of order 13.

**Lemma 3.8.** The color diagrams \( D(n, k) \) and \( D'(n, k) \) are monotonic.

**Lemma 3.9.** Let \( P \) be an increasing prefix of \( D(n, k) \) or \( D'(n, k) \). Then \( P \) is monotonic.

**Theorem 3.10.** Let \( G \) be a split graph. If \( \Delta(G) \) is even, then \( G \) is type one.

**Proof:** First, we consider the ICLS \( M(k) \), where \( \Delta(G) + 1 = 2k - 1 \). Let \( U = \{ u_1, u_2, \ldots, u_n \} \), \( V = \{ v_1, v_2, \ldots, v_r \} \). By Lemma 2.1, without loss of generality, we may assume that \( \text{deg}(u_i) = \Delta(G) \) for each \( i = 1, 2, \ldots, n \). Now we will use \( M(k) \) to obtain a total coloring \( \pi \) for \( G \) which uses \( \Delta(G) + 1 \) colors (entries of \( M(k) \)). We consider two cases.

**Case 1:** \( n \) is even. For the vertices of \( G \) and the edges of \( G[U] \), let \( \pi(u_i) = m_{i,i}, i = 1, 2, \ldots, n; \pi(v_j) = m_{1,n} (> n), j = 1, 2, \ldots, r; \) and \( \pi(u_iv_j) = m_{4,j}, 1 \leq i, j \leq n \). To complete the total coloring \( \pi \), the edges incident to \( u_i \) in \( G\setminus E(G[U]) \) have to use colors in \( S_{u_i} = \{ m_{4,j} | n+1 \leq j \leq 2k-1 \} \) for each \( 1 \leq i \leq n \). The color diagram \( C(n, k) \) is an arrangement of edge coloring scheme \( S = \{ S_{u_i} \}_{i=1}^n \). Since \( C(n, k) \) is monotonic, by Lemma 3.5, there is an edge coloring \( \pi_1 \) of \( G\setminus E(G[U]) \) which fulfills \( S \). Let \( \pi(u_iv_j) = \pi_1(u_iv_j) \) for all \( u_iv_j \in G\setminus E(G[U]) \). Since color \( m_{1,n} \) is larger than \( n \) and does not occur in \( C(n, k) \), \( \pi \) is a total coloring using \( \Delta(G) + 1 \) colors.

**Case 2:** \( n \) is odd. We note here that we cannot use the method of coloring used in Case 1 to color the vertices in \( V \) because \( m_{1,n} < n \). First, the vertices of \( U \) and the edges of \( E(U) \) will be colored as in Case 1. Without loss of generality, let the neighbor set of \( v_1 \) be \( \{ u_1, u_2, \ldots, u_{\text{deg}(v_1)} \} \). Let

142
\( \pi(v_1) = n + 1 \) and \( \pi(v_j) = m_{1,n+1} (> n) \). Now if we can color the edges in \( E(G) \setminus E(G[U]) \) properly, then we have the proof. First, for each \( i \geq 1 \), if \( u_i v_1 \in E(G) \), let \( \pi(u_i v_1) = m_{i,n+1} (\neq n+1) \). For the rest of the edges, the remaining colors that can be used for \( u_i \) are \( S_{u_i} = \{ m_{i,j} \, | \, n+2 \leq j \leq 2k-1 \} \) for \( 1 \leq i \leq \deg(v_1) \) and \( S_{u_i} = \{ m_{i,j} \, | \, n+1 \leq j \leq 2k-1 \} \) otherwise. It is easy to see that the edge coloring scheme \( S = \{ S_{u_i} \}_{i=1}^{n} \) can be arranged as a suffix of \( \mathcal{D}(n,k) \). Thus, by Lemma 3.6 and Lemma 3.7, \( S \) can be arranged into a monotonic color diagram. By Lemma 3.5, there is an edge coloring \( \pi_1 \) of \( \{ u_i v_j \mid j > 1 \text{ and } u_i v_j \in E \} \) which fulfills scheme \( S \). Let \( \pi(u_i v_j) = \pi_1(u_i v_j) \) for all \( u_i v_j \in E \) and \( j > 1 \). We obtain a total coloring of \( G \) using \( \Delta(G) + 1 \) colors.

The chromatic index of the split graph is quite different.

**Theorem 3.11.** Let \( G \) be a split graph. If \( \Delta(G) \) is odd, then \( G \) is class one.

**Proof:** Let \( \Delta(G) = 2k - 1 \) for some positive integer \( k \). Without loss of generality, by Lemma 2.1, we may assume that \( \deg(u) = \Delta(G) \) for all \( u \in U \). Let \( h = \min \{ x \mid |\bigcup_{i=1}^{x} N(v_i)| \geq \frac{n}{2} \} \) and \( |\bigcup_{i=1}^{h} N(v_i)| = b \), where \( N(v_i) \) denote the neighborhood of \( v_i \) in \( G \). Let \( B \) be the bipartite subgraph of \( G \) with parties \( U \) and \( V \) and edge set \( E(G) \setminus E(G[U]) \). Let \( H_1 = B[U \cup \{ v_{h+1}, \ldots, v_{r} \}] \) and \( H_2 = B[U \cup \{ v_1, v_2, \ldots, v_h \}] \). We can order the vertices of \( U \) such that \( \deg_{H_2}(u_i) \geq \deg_{H_2}(u_{i-1}) \) for each \( i = 1, 2, \ldots, n - 1 \). Moreover, we may assume that if \( \deg_{H_2}(u_i) = \deg_{H_2}(u_j) \) and \( u_i v_h \in E(G) \), then \( i > j \). As a result of this assumption, \( u_i v_h \in E(G) \) for all \( n/2 \leq i \leq b \), since for \( n/2 \leq i \leq b \), \( d_{H_2}(u_i) = 1 \) and \( u_i \) is joined to \( v_h \).

We will use the ICLS \( M(k) \), of order \( 2k - 1 \), to obtain an edge coloring \( \varsigma \) for \( G \) of \( 2k - 1 \) colors. First, the edges of \( G[U] \) will be colored in a manner similar to that used in the proof of Theorem 3.10, i.e., \( \varsigma(u_i u_j) = m_{i,j} \), for \( 1 \leq i \neq j \leq n \). For the edges in \( E(G) \setminus E(G[U]) = E(H_1) \cup E(H_2) \), we will use a process similar to that used in Theorem 3.10 to color them except that some adjustments must be made. Mainly, this is because all \( m_{i,i}, 1 \leq i \leq n \), should be used in the edge coloring in order to show that \( G \) is class one.

We will color edges in \( E(H_1) \) and \( E(H_2) \) separately. For edges in \( E(H_1) \), consider a color diagram \( C_1 = (R_{u_1}, \ldots, R_{u_n}) \) such that \( R_{u_i} = (m_{i,n+1}, \ldots, m_{i,\deg_{H_1}(u_i)+n}) \) for \( 1 \leq i \leq b \) and \( R_{u_i} = (m_{i,i}, m_{i,n+1}, \ldots, m_{i,2k-1}) \) for \( b + 1 \leq i \leq n \). Since \( b > n/2 \) and \( \deg_{H_1}(u_1) \leq \deg_{H_1}(u_2) \leq \cdots \leq \deg_{H_1}(u_n) \), \( C_1 \) is a suffix of an increasing prefix of \( \mathcal{D}(n,k) \). By Lemma 3.6 and Lemma 3.9, \( C_1 \) is monotonic. Thus, by Lemma 3.5, we have an edge coloring \( \varsigma_1 \) which fulfills \( C_1 \), for edges in \( E(H_1) \).

Next, we will find an edge coloring \( \varsigma_2 \) for edges in \( E(H_2) \). First we color edges incident to \( v_h \). Let \( \varsigma_2(u_i v_h) = m_{i,i} \) for \( u_i v_h \in E(G) \). For the re-
maining edges to be colored, consider a color diagram \( C_2 = (A_{u_1}, \ldots, A_{u_b}) \) where for \( i > b, \ A_{u_i} \) is an empty array, and for \( i \leq b, \ A_{u_i} = (m_{i,2k-1}, \ldots, m_{i,\deg_{H_1}(u_i)+n+1}) \) if \( u_iv \in E(G) \), otherwise we have \( A_{u_i} = (m_{i,i}, m_{i,2k-1}, \ldots, m_{i,\deg_{H_1}(u_i)+n+1}) \). By the assumption concerning the ordering of the \( u_i \)'s, only \( m_{i,i}, i < n/2 \), may occur in \( C_2 \), and thus \( C_2 \) is a suffix of an increasing prefix of \( D'(n, k) \). By Lemma 3.6 and Lemma 3.9, \( C_2 \) is monotonic. By Lemma 3.5 again, we have an edge coloring for \( B[U \cup \{v_1, \ldots, v_{n-1}\}] \) which fulfills \( C_2 \). Together with the edge coloring of edges incident to \( v_h \), we have an edge coloring \( \varsigma_2 \) for \( E(H_2) \).

\[
\begin{array}{c}
\text{Figure 2: An illustration of color diagrams } C_1 \text{ and } C_2.
\end{array}
\]

As shown in Figure 2, the colors used by \( \varsigma_1 \) and \( \varsigma_2 \) for edges incident to \( u_i \) are exactly \( m_{i,i}, m_{i,n+1}, \ldots, m_{i,2k-1} \) for each \( 1 \leq i \leq n \). Let \( \varsigma(u_iv_j) = \varsigma_1(u_iv_j) \) for \( u_iv \in E(H_1) \) and \( \varsigma(u_iv_j) = \varsigma_2(u_iv_j) \) for \( u_iv_j \in E(H_2) \). We obtain an edge coloring for \( G \) using \( \Delta(G) \) colors.

\( \square \)

Last, we provide an example to illustrate our proof of Theorem 3.11.
(Figure 1 will be used in this example.)

Example: Let $G$ be a split graph of order 19 such that $U = \{u_1, u_2, \ldots, u_{10}\}$ and $V = \{v_1, v_2, \ldots, v_{10}\}$, and let the adjacent matrix of $G$, $A(G)$, be as in Figure 3. Then $\Delta(G) = 13$, $n = 9$, $h = 3$, and $b = 6$. Thus we can use $M(7)$ in Figure 1 to find an edge coloring of $G$. In Figure 4, an entry with a pair $(i, j)$ and a number $x$ below the pair indicates that the edge $u_i, v_j$ is colored with $x$.

$$A(G) = \begin{bmatrix} J_9 - J_9 \\ B^t \\ O \end{bmatrix}$$

$$B = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

**Figure 3:** Adjacency matrix of $G$.

**Figure 4:** An edge coloring of $G$

4. Remarks

To date, we have not succeeded in using the technique employed in this paper to find either the total coloring or the edge coloring of a graph. We
hope to obtain further results using this technique. Finally, we would like to express our thanks to Dr. A.J.W. Hilton, Dr. H. Hind, and Dr. H.P. Yap for their helpful comments.

References


