A New Construction for a Critical Set
in Special Latin Squares

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Abstract

A critical set is a partial latin square which is completable to a latin square
and omitting any entry of the partial latin square destroys this property. The
size of a critical set is the number of entries in the partial latin square, and
a critical set with minimum (maximum) size is called a minimal (maximal)
critical set. In this paper, we study the minimal and maximal critical sets, and
some results are obtained. Mainly, we prove that every minimal critical set in a
latin square of order \(n\) has size at least \(n + 1\). Also, we show that the maximal
critical set of a latin square of order \(2^n - 1\) contains at least \(4^n - 3^n - 2^{n+1} + 3\)
entries.

1 Introduction

A latin square of order \(n\) is an \(n \times n\) array with entries in \(N = \{1, 2, \ldots, n\}\) such that
each element of \(N\) occurs in each row and each column exactly once. A partial latin
square of order \(n\) is an \(n \times n\) array such that each element of \(N\) occurs at most once in
each row and at most once in each column. A critical set in a latin square \(L\) of order
\(n\), is a set \(A = \{(i, j; k) \mid i, j, k \in N\}\) such that, \(L\) is the only latin square of order
\(n\) which has element \(k\) in position \((i, j)\) for each \((i, j; k) \in A\), and no proper subset
of \(A\) satisfies the above property. The size of a critical set \(A\) is \(|A|\) and a critical set
with minimum (maximum) size is called a minimal (maximal) critical set.

In [3], Lemma 2.3, Curran and van Rees showed that, if you take the ordered
triples \((x, y; z)\) of a critical set, then the \(i^{th}\) component of these triples, \(i = 1, 2, 3,\)
must cover at least \(n - 1\) of the values 1, 2, \ldots, \(n\), thereby showing that the size of
the minimal critical set is no less than \(n - 1\). Later, in [4], Lemma 3.1, Donovan et.
al. improved this bound to \(n\) for each \(n \geq 4\). In this paper, we shall prove that the
size of a minimal critical set is at least \(n + 1\) for each \(n \geq 5\). Note that this result is
also obtained in [2] by a different method.

As to the maximal critical sets, it was shown by Stinson and van Rees [6] that the
maximal critical set of a latin square of order \(2^n\) contains at least \(4^n - 3^n\) entries. In

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section 3, we shall use a special construction to prove that the maximal critical set of a latin square of order $2^n - 1$ contains at least $4^n - 3^n - 2^{n+1} + 3$ entries.

2 Minimal critical set

Two latin squares are isotopic, if one can be transformed onto the other by permuting rows, permuting columns and renaming the entries. Thus two partial latin squares $L$ and $L'$ are isotopic if there exists an ordered triple $(\alpha, \beta, \gamma)$ of permutations such that $L(i, j)$, the $(i, j)$ position in $L$, is $k$ if and only if $L'(i\alpha, j\beta) = k\gamma$. In particular, two critical sets $C$ and $C'$ are said to be isotopic if they are isotopic as two partial latin squares. Then the following result is easy to see.

Lemma 2.1 ([4]) Let $C$ be a critical set in a latin square $L$. Let $(\alpha, \beta, \gamma)$ be an isotopism from the critical set $C$ onto $C'$. Then $C'$ is a critical set in the latin square $L'$ obtained from $L$ by applying the isotopism $(\alpha, \beta, \gamma)$.

In order to obtain the lower bound for the minimal critical sets in a latin square, we shall rely on the following result.

Theorem 2.2 (L. D. Anderson [1]) A partial latin square of order $n$ with size at most $n + 1$ can be completed to a latin square if it doesn’t contain a partial latin square as in Figure 1 (Appendix).

Theorem 2.3 Let $A$ be a critical set in a latin square of order $n \geq 5$, then $|A| \geq n + 1$.

Proof. Assume that $A$ is a critical set and $|A| \leq n$. We shall claim that there exists a position not in $A$ which we can choose two elements in $N = \{1, 2, \ldots, n\}$ to fill in respectively, and both can be completed to a latin square. This implies that $A$ is not a critical set and thus $|A|$ must be at least $n + 1$.

Let $L$ be a latin square of order $n$. If $n$ cells which jointly contain at least $n - 1$ different symbols occupy at least $n - 1$ rows and $n - 1$ columns of $L$, then there exists at most one row and/or column which contains two of the cells. In this one row and/or column (if such exists), the entries are distinct so the entries in the cells of the remaining $n - 2$ rows/columns use at least $n - 3$ distinct symbols.

If they use $n - 2$ distinct symbols, then they form a partial transversal of $n - 2$ cells. If they use $n - 3$ distinct symbols, then in the exceptional row/column which contains two of the cells, there is a cell whose entry is distinct from the $n - 3$ symbols, so again we can find a partial transversal of length $n - 2$.

By the result of Donovan et. al. in [4], we may suppose that $|A| = n$, and also the arguments are up to isotopisms. Since the position in $A$ will occur in at least $n - 1$ rows and $n - 1$ columns, and the entries occur in $A$ should contain at least $n - 1$ elements of $N$. By above discussion, we may assume that the triples in $A$ by $(i, i; i)$, for $i = 1, 2, \ldots, n - 2$, $(n - 1, n - 1; x)$ and $(a, b; y)$ where $x = n - 1$ or $y = n - 1$. Now let $(c, d)$ be a position not in $A$ and $c \neq n - 1, d \neq n - 1$. If $n \geq 5$, then there are least
two elements $t_1$ and $t_2$ in $N$ which do not occur in the $c^{th}$ row and $d^{th}$ column of the partial latin square. Now let $(c, d)$ be filled with $t_1$ and $t_2$ respectively, and we obtain $A_1$ and $A_2$ respectively. Clearly, $A_1$ and $A_2$ are two distinct partial latin squares of size $n + 1$, and $A_1$ and $A_2$ do not contain any type of partial latin square in Figure 1. This implies that by Theorem 2.2, $A_1$ and $A_2$ can be completed respectively. Thus we have the claim and the proof.

3 Maximal critical set

A latin square $L = [L(i, j)]$ of order $n$ is idempotent if $L(i, i) = i$ for each $i$, unipotent if $L(i, i) = c$ where $c$ is a constant, and commutative if $L(i, j) = L(j, i)$ for all $i, j$. A quasigroup satisfying the Steiner identities: $x^2 = x$, $x(xy) = y$ and $(yx)x = y$, is called a Steiner quasigroup. From the definition of Steiner quasigroup, we know that a latin square obtained by a Steiner quasigroup is idempotent and commutative. Let $(P, \ast)$ be a Steiner quasigroup of order $v \geq 3$ and $p, q \in P$. Then $(P, \ast)$ contains the following subquasigroup.

<table>
<thead>
<tr>
<th>*</th>
<th>p</th>
<th>q</th>
<th>p \ast q</th>
</tr>
</thead>
<tbody>
<tr>
<td>p</td>
<td>p</td>
<td>p \ast q</td>
<td>q</td>
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<tr>
<td>q</td>
<td>p \ast q</td>
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<tr>
<td>p \ast q</td>
<td>q</td>
<td>p</td>
<td>p \ast q</td>
</tr>
</tbody>
</table>

The following result is easy to see.

Lemma 3.1 Let $L$ be a latin square obtained by a Steiner quasigroup. Then any tow distinct elements in $L$ are contained in exactly one latin subsquare of order 3.

Stinson and von Rees [6] defined the double of a latin square $L$ and a critical set $C$ in $L$, $2 \ast L$ and $2 \ast C$, as follows:

$$2 \ast L = \begin{bmatrix} L_1 & L_2 \\ L_2 & L_1 \end{bmatrix} \quad 2 \ast C = \begin{bmatrix} L_1 & C_2 \\ C_2 & L_1 \end{bmatrix}$$

where $L_i$ and $C_i$ ($i = 1, 2$) is a copy of $L$ and $C$ respectively, with every symbol $x$ in $L$ replaced by $x_i$. At the same time they showed that if $C$ is a critical set in $L$, then $2 \ast C$ is a critical set in $2 \ast L$. Thus a critical set of the latin square representing 2-group of order $2^n$ can be constructed by above method. Therefore we have the following theorem.

Theorem 3.2 ([6]) The maximal critical set of latin squares of order $2^n$ contains at least $4^n - 3^n$ elements.

Insteaded of prolongation [5], we will use a compression to obtain a latin square of order $2^n - 1$ from the latin square of order $2^n$.

Let $L = [L(i, j)]$ be the latin square representing 2-group of order $2^n$ based on $\{1, 2, \ldots, 2^n\}$. Let $M = [M(i, j)]$ be an array constructed by replacing the diagonal
entries of $L$ with the 1\textsuperscript{st} row of $L$, and taking away its 1\textsuperscript{st} column and 1\textsuperscript{st} row. That is for each $i, j$ in \{1, 2, \ldots, 2^n - 1\}

$$M(i, i) = L(1, i + 1), \text{ and}$$

$$M(i, j) = L(i + 1, j + 1), \text{ for } i \neq j.$$ 

Then $M$ has the following properties:

(1) $M$ is an idempotent latin square of order $2^n - 1$ based on \{2, 3, \ldots, 2^n\}. (Since $L$ is unipotent and commutative, and its 1\textsuperscript{st} row is 1, 2, 3, \ldots, 2^n)

(2) $M$ determines a Steiner quasigroup.

Let $x$ and $y$ be two distinct elements in \{2, 3, \ldots, 2^n\}. If $x \ast y = z$ in $M$, then $L$ contains the following partial latin square.

\[
\begin{array}{cccc}
1 & x & y & z \\
 x & 1 & z & \\
y & z & 1 & \\
z & & & 1 \\
\end{array}
\]

Since $L$ is the latin square representing 2-group, any two entries filled the same element are contained in a latin subsquare of order 2. Thus $x \ast z = y$ and $y \ast z = x$. Therefore $M$ is a Steiner quasigroup.

The following example explains the above idea.

![Table L and M]

$L$ is the latin square representing 2-group of order 8 and $M$ is a latin square of order 7. By Theorem 3.2, we can obtain a critical set $C(L)$ of $L$ as follows.

![Table C(L) and C(M)]

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Corresponding to \( C(L) \) we expect \( C(M) \) is a critical set of \( M \). This is a result of the following theorem.

**Theorem 3.3** Let \( L \) be a latin square of order \( 2^n \) representing 2-group and \( C(L) \) be the critical set of \( L \) constructed as in Theorem 3.2. Let \( M \) be the latin square obtained from \( L \) by using a compression as explained above. Then \( C(M) = \{(i, j; M(i, j))|i, j \in \{1, 2, \ldots, 2^n - 1\}, \text{ and } (i + 1, j + 1; L(i + 1, j + 1)) \in C(L)\} \) is a critical set of \( M \).

**Proof.** Since \( C(L) \) is a critical set of \( L \), by construction it is easy to see that \( C(M) \) can be completed to \( M \) and also the completion is unique. Now we claim that removing any entry of \( C(M) \) will destroy this property. First, if an entry \((i, j; u)\) of \( C(M) \) not in diagonal is removed, correspondingly an entry in \( C(L) \) not in the first row is removed. By the fact that \( C(L) \setminus \{(i + 1, j + 1; L(i + 1, j + 1))\} \) can not be completed uniquely, \( C(M) \setminus \{(i, j; u)\} \) can not be completed uniquely either. Finally, if we remove any entry \((i, i; x)\) of \( C(M) \), then since \( M \) determines a Steiner quasigroup \((M, *)\) there exists a \( 3 \times 3 \) subsquare of \( M \) determined by \( x \) and \( 2^n \), which intersects \( C(M) \) at exactly one entry \((x, 2^n; x \ast 2^n)\). This implies that \( C(M) \setminus \{(i, i; x)\} \) is not uniquely completable. A minimal critical set of a latin square of order 3 contains at least 2 entries. Therefore we have the proof. ⬤

By direct counting, we obtain the following result for the size of the maximal critical sets.

**Corollary 3.4** The maximal critical set of latin squares of order \( 2^n - 1 \) contains at least \( 4^n - 3^n - 2^{n+1} + 3 \) elements.

**References**


Appendix

Type 1 (1 ≤ z < n)

Type 2 (1 ≤ z < n)

Type 3 (1 ≤ z < n)

The Noncompletable Partial Latin Squares of Side n with n Nonempty Cells.

Type 4 (n ≥ 3)

Type 5 (n ≥ 3)

Type 6 (n ≥ 3)

Type 7 (n ≥ 4)

Type 8 (n ≥ 4)

Type 9 (n ≥ 4)

Type 10 (n ≥ 5)

Type 11 (n ≥ 5)

Type 12 (n ≥ 5)

The Noncompletable Partial Latin Squares of Side n with n + 1 Nonempty Cells.