Connectivity of Cages

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ABSTRACT

A (k; g)-graph is a k-regular graph with girth g. Let f(k; g) be the smallest integer ν such there exists a (k; g)-graph with ν vertices. A (k; g)-cage is a (k; g)-graph with f(k; g) vertices. In this paper we prove that the cages are monotonic in that f(k; g1) < f(k; g2) for all k ≥ 3 and 3 ≤ g1 < g2. We use this to prove that (k; g)-cages are 2-connected, and if k = 3 then their connectivity is k. © 1997 John Wiley & Sons, Inc.

1. INTRODUCTION

All graphs in this note are simple. The length of a shortest odd or even cycle in a graph G is called the odd girth or the even girth of G, respectively. Throughout this paper let g = g(G) denote the smaller of the odd and even girths of G (so g is the girth of G), and let h = h(G) denote the larger; then the girth pair of G is defined to be (g, h). A k-regular graph with girth pair (g, h) is called a (k; g, h)-graph. For any k ≥ 1 and any g ≡ h (mod 2) with 3 ≤ g < h, let f(k; g, h)

*This research is supported by ONR Grant N000014-95-0769.

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CCC 0364-9024/97/020187-05
denote the smallest integer $\nu$ such that there exists a $(k; g, h)$-graph with $\nu$ vertices. Similarly, a $k$-regular graph with girth $g$ is called a $(k; g)$-graph, and let $f(k; g)$ denote the smallest integer $\nu$ such that there exists a $(k; g)$-graph with $\nu$ vertices; a $(k; g)$-graph with $f(k; g)$ vertices is called a cage. Cages have been studied widely since introduced by Tutte in 1947 [3]; see [4] for a survey referring to 70 publications.

Several interesting questions concerning girth pairs of graphs remain open. For example, it is clear that $f(k; g) \leq f(k; g, h)$, and this inequality may be strict; for example, the $(k; 4)$-cage is $K_{2k}$ [4], so contains no 5-cycles, so in this case $f(k; 4) < f(k; 4, 5)$. Related to this observation is a conjecture of Harary and Kovacs [2] who believe that if $g$ is odd then $f(k; g) = f(k; g, g+1)$. But whether $f(k; g, h) \leq f(k; h)$ remains unknown. Harary and Kovacs proved [2] that $f(k; h - 1, h) \leq f(k; h)$. They also conjectured that all $(k; g, h)$-graphs of order $f(k; g, h)$ are 2-connected. In this paper we prove the related conjecture that cages are 2-connected. Our proofs rely on knowing that cages are monotonic in the sense that $f(k; g_1) < f(k; g_2)$ for all $g_1 < g_2$. While this may be known to some, we can find no reference to the result, so a proof is included here. For any undefined terminology, see [1].

2. MONOTONICITY AND CONNECTIVITY OF CAGES

There have been many papers that find bounds on $f(k; g)$ (see [4] for a survey). We begin by considering $f(k; g)$, proving that cages are monotonic, a result that will also be of use in considering the connectivity of cages.

Theorem 1. For all $k \geq 3$ and $3 \leq g_1 < g_2$, $f(k; g_1) < f(k; g_2)$.

Proof. It suffices to show that if $k, g \geq 3$ then $f(k; g) < f(k; g + 1)$. So let $G$ be a $(k; g + 1)$-graph with $f(k; g + 1)$ vertices.

Suppose $k$ is even. Let $C$ be a cycle of length $g + 1$ in $G$ containing the edges $uv_1$ and $uv_2$. Let $N_G(u) = \{v_1, \ldots, v_k\}$ be the neighborhood of $u$ in $G$, and let $E' = \{v_1v_2, v_3v_4, \ldots, v_{k-1}v_k\}$. Let $G'$ be the component of $G - u + E'$ that contains $v_1$. Since $g + 1 \geq 4$, $N_G(u)$ is an independent set of $G$, so $E' \cap E(G) = \emptyset$, and so $G'$ is a simple graph. Clearly $G'$ contains the cycle $(C - u) + v_1v_2$ of length $g$. Also, if $C'$ is a cycle in $G'$ then: if $E' \cap E(C') = \emptyset$ then $C'$ is a cycle in $G$; and if $E' \cap E(C') \neq \emptyset$ then let $P$ be a $(v_1, v_1)$-path that is a subgraph of $C'$ with $E(P) \cap E' = \emptyset$, so $P + \{uv_1, w_1\}$ is a cycle in $G$, so $C'$ has length at least $g$ (since $C'$ contains $P$ and at least one edge in $E'$). So $G'$ has no cycles of length less than $g$, and is therefore a $(k; g)$-graph with at most $f(k; g + 1) - 1$ vertices, so $f(k; g) < f(k; g + 1)$.

Suppose $k$ is odd. Let $C$ be a cycle of length $g + 1$ in $G$ containing $uv_1$ and $uv_2$. Let $N_G(u) = \{v_1, \ldots, v_{k-1}, w\}$. Clearly $w \notin V(C)$, for if $C$ is the cycle $(u, v_2, \ldots, x_1, w, x_2, \ldots, v_1)$ then $(u, v_2, \ldots, x_1, w)$ is a cycle of length less than the girth of $G$. Let $N_G(u) = \{x_1, \ldots, x_{k-1}, u\}$. Let $G'$ be the component of $(G - \{u, w\}) + \{v_{2i-1}v_{2i}, x_{2i-1}x_{2i} | 1 \leq i \leq (k-1)/2\}$ that contains $v_1$. Since $g + 1 \geq 4$, $N_G(u)$ and $N_G(w)$ are independent sets of $G$, so $G'$ is simple. Clearly $C - u + v_1v_2$ is a cycle in $G'$ of length $g$, and (as in the previous case) no cycle in $G'$ has length less than $g$. Therefore $G'$ is a $(k; g)$-graph with at most $f(k; g + 1) - 2$ vertices, so $f(k; g) < f(k; g + 1)$.

We can now use Theorem 1 to prove the following result.

Theorem 2. All $(k; g)$-cages are 2-connected.
Prove. Suppose that \( G \) is a connected \((k, g)\)-graph that contains a cut vertex \( u \). Let \( C_1, \ldots, C_w \) be the components of \( G - u \), with \( |V(C_i)| \leq |V(C_j)| \) for \( 1 \leq i < j \leq w \). Clearly
\[
d_{C_1}(v_1, v_2) \geq g - 2 \quad \text{for all } v_1, v_2 \in V(C_1) \cap N_G(u).
\]
Let \( C' \) be a copy of \( C_1 \) with \( V(C') \cap V(C_1) = \emptyset \), and let \( f \) be an isomorphism between \( C_1 \) and \( C' \). Form a new graph from the union of \( C_1 \) and \( C' \) by joining each \( v \in V(C_1) \cap N_G(u) \) to \( f(v) \) with an edge.

Clearly \( H \) is \( k \)-regular, and has fewer vertices than \( G \) (since \( |V(C')| \leq |V(C_2)| \) and \( u \not\in V(H) \)). Also, by (1), any cycle in \( H \) containing an edge \( v f(v) \) has length at least \( 2(g - 2) + 2 = 2g - 2 \), so \( H \) has girth at least \( \min\{g, 2g - 2\} = g \). Therefore by Theorem 1, \( G \) is not a \((k, g)\)-cage, and the result follows.

3. FURTHER RESULTS

While it is good to know that cages are 2-connected, we believe that their connectivity is much higher. Indeed, we are bold enough to make the following conjecture.

Conjecture. All simple \((k, g)\)-cages are \( k \)-connected.

In support of this conjecture, we now prove the following result.

Theorem 3. All cubic cages are 3-connected.

Prove. Suppose \( G' \) is a \((3, g)\)-cage. By Theorem 2, \( G' \) has connectivity at least 2. Suppose \( G' \) has connectivity 2. The following construction of a graph \( G \) is depicted in Figure 1.

Since \( G' \) is a cubic cage, \( G' \) has an edge-cut consisting of two edges, say \( e \) and \( f \). Let \( H' \) and \( W' \) be the two components of \( G' - \{e, f\} \), let \( e = x_0y_0 \) and \( f = x_1y_1 \), where \( \{x_0, x_1\} \subseteq H' \) and \( \{y_0, y_1\} \subseteq W' \). Let \( d_{W'}(y_0, y_1) = d' \leq d_{H'}(x_0, x_1) = D \). Let \( P = (w_0 = y_0, w_1, w_2, \ldots, w_d = y_1) \) be a shortest \((y_0, y_1)\)-path in \( W' \), let \( Q' = (h_0 = x_0, h_1, h_2, \ldots, h_D = x_1) \) be a shortest \((x_0, x_1)\)-path in \( H' \) and let \( Q = (h_0, h_1, \ldots, h_{d-1}) \) be the \((x_0, h_{d-1})\)-subpath of \( Q' \). For each \( i \in \{0, 1\} \) let \( z_i \) be the unique neighbor of \( y_i \) in \( W' \) that is not in \( P \). Let \( R \) be the path \((z_0, x_0, w_1, h_1, w_2, h_2, \ldots, w_{d-1}, h_{d-1}) \). Let \( H = H' - E(Q) \) and let \( L = (W' - E(P)) - \{y_0, y_1\} \). Let \( G = (H \cup W \cup R) + \{x_1z_1\} \) (see Fig. 1).

Clearly \( G \) is a cubic graph with \(|V(G')| - 2\) vertices. We now show that \( G \) has girth at least \( g \), so the result will then follow from Theorem 1 which will contradict \( G' \) being a \((3, g)\)-cage.

Any cycle in \( G \) that is also in \( G' \) clearly has length at least \( g \). Any cycle in \( G \) that is not in \( G' \) contains at least two edges in \( E(R) \cup \{x_1z_1\} \); let \( C \) be a cycle containing exactly two such edges, say \( c_1 \) and \( c_2 \). We consider several cases.

Case 1. Suppose \( c_1 = x_0z_0 \) and \( c_2 = h_{i-1}w_i \) or \( h_iw_i \) with \( 1 \leq i \leq d - 1 \).

Let \( P_1 \) be a shortest \((z_0, w_i)\)-path in \( W \). Then \( P_1 \) is a path in \( W' \). Let \( P_2 \) be the \((y_0, w_i)\)-subpath of \( P \); so \( P_2 \) has length \( i \). Then clearly \((P_1 \cup P_2) + y_0z_0 \) contains a cycle of length at most \( i + 1 + d_{W}(z_0, w_i) \). Since \((P_1 \cup P_2) + y_0z_0 \) is a subgraph of \( G' \), \( i + 1 + d_{W}(z_0, w_i) \geq g \). For each \( i \in \{i - 1, i\} \), \( d_{H}(x_0, h_i) \geq d_{H}(x_0, h_i) = i - 1 \), so \( C \) has length at least \( d_{H}(x_0, h_l) + d_{W}(z_0, w_i) + 2 \geq i - 1 + g - (i + 1) + 2 = g \).

Case 2. Suppose \( c_1 = x_0z_0 \) and \( c_2 = x_1z_1 \).

Let \( P_1 \) be a shortest \((z_0, z_1)\)-path in \( W \). Then \((P_1 \cup P) + \{y_0z_0, y_1z_1\} \) contains a cycle, and this cycle has length at most \( d + 2 + d_{W}(z_0, z_1) \). Since this cycle is also a subgraph of
$G', d + 2 + d_W(z_0, z_1) \geq g$. Clearly $d_H(x_0, x_1) \geq d_H'(x_0, x_1) = D$. Therefore $C$ has length at least $d_H(x_0, x_1) + d_W(z_0, z_1) + 2 \geq D + g - (d + 2) + 2 \geq g$.

Case 3. Suppose $e_1 = h_{i-1}w_i$ or $h_iw_i$ and $e_2 = h_{j-1}w_j$ or $h_jw_j$, with $1 \leq i \leq j \leq d - 1$.

If $i = j$ then we can assume $e_1 = h_{i-1}w_i$ and $e_2 = h_iw_i$, so $C - \{e_1, e_2\} + h_{i-1}h_i$ is a cycle in $G'$, and so has length at least $g$. Therefore $C$ has length at least $g + 1$.

If $i < j$ then let $P_1$ be a shortest $(w_i, w_j)$-path in $W$. Since $P_1 + \{w_lw_{l+1}|i \leq l < j\}$ contains a cycle in $G'$, $P_1$ has length at least $g - (j - i)$. Also, for each $l_1 \in \{i - 1, i\}$ and each $l_2 \in \{j - 1, j\}$, $d_H(h_{l_1}, h_{l_2}) \geq d_H'(h_i, h_{j-1}) = j - 1 - i$. So $C$ has length at least $g - (j - i) + (j - 1 - i) + 2 = g + 1$.

Case 4. Suppose $e_1 = h_{i-1}w_i$ or $h_iw_i$ with $1 \leq i \leq d - 1$ and $e_2 = x_1z_1$.

As in the previous case $d_W(w_i, z) \geq g - (d + 1 - i)$, and for each $l \in \{i - 1, i\}$ $d_H(h_l, x_1) \geq d_H'(h_i, x_1) = d - i$. Therefore $C$ has length at least $g - (d + 1 - i) + (d - i) + 2 = g + 1$.

Thus in every case, if $C$ contains exactly two edges in $R$ then $C$ has length at least $g$. If $C$ contains more than two edges in $R$ then it follows even more easily that $C$ has length at least $g$, so the result is proved.
ACKNOWLEDGMENTS

The authors wish to thank a referee for the shorter proof of Theorem 2 that appears in this paper.

References


Received October 18, 1995