2-colouring \(\{C_3, C_4\}\)-designs

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A design \((V, B)\) with point set \(V\) and collection of blocks \(B\) is said to have a blocking set \(S\) if there exists a non-empty proper subset \(S\) of \(V\) such that every block in \(B\) meets \(S\), and no block in \(B\) lies entirely in \(S\). Equivalently, we may colour the points in \(V\) with two colours, say red if a point lies in \(S\) and blue if it lies in \(V\setminus S\), so that no block is monochromatic. If this is possible, the design is said to be 2-colourable. It is well-known that no Steiner triple system (of order greater than 3) can be 2-coloured. A very recent paper on 2-colouring cycle systems, which includes many useful references, is [2].

For the purpose of this note, we may consider an \(m\)-cycle system of order \(n\) to be an edge-disjoint decomposition of \(K_n\) into cycles of length \(m\). If a cycle of length \(m\) has edges \(\{a_i, a_{i+1}\}\) for \(1 \leq i \leq m-1\) and \(\{a_1, a_m\}\), then we write the cycle as \(\langle a_1, a_2, \ldots, a_m \rangle\) or \(\langle a_1, a_m, a_{m-1}, \ldots, a_2 \rangle\) or any cyclic shift of these. A \(\{C_3, C_4\}\)-design of order \(n\) is then an edge-disjoint decomposition of \(K_n\) into copies of \(C_3\) and \(C_4\). It is well-known (see [1]) that a \(\{C_3, C_4\}\)-design of order \(n\) with \(p\) copies of \(C_3\) and \(q\) copies of \(C_4\) exists if and only if \(n\) is odd and \(3p + 4q = \binom{n}{3}\). In fact it is easy to verify that if \(n \equiv i \pmod{8}\) where \(i\) is odd, then \(p = 4t + (i-1)/2\), for

\[ t = 0, 1, \ldots, \left\lfloor \frac{n(n-1)}{24} \right\rfloor. \] We shall say that such a \( \{C_3, C_4\} \)-design is of type \( (p, q) \) if it contains \( p \) copies of \( C_3 \) and \( q \) copies of \( C_4 \).

Our aim in this note is to prove the following.

**Theorem 1** Let \( n \) be odd. Then there exists a \( \{C_3, C_4\} \)-design of order \( n \) and type \( (p, q) \) which can be 2-coloured if and only if \( 3p + 4q = \binom{n}{2} \) and \( p \leq (n-1)/2 \).

**Proof:** Certainly the requirement that \( 3p + 4q = \binom{n}{2} \) is necessary for any \( \{C_3, C_4\} \)-design of type \( (p, q) \). So we start by showing the necessity of \( p \leq (n-1)/2 \) in a 2-coloured \( \{C_3, C_4\} \)-design of type \( (p, q) \).

Suppose we have a blocking set \( S \) of cardinality \( s \) in a \( \{C_3, C_4\} \)-design of order \( n \) and type \( (p, q) \). Then certainly \( n \) must be odd, and counting the number of edges,

\[ 3p + 4q = \frac{n(n-1)}{2}. \]

Now counting the \( s(n-s) \) edges of \( K_n \) that join \( S \) with the other \( n-s \) vertices, each \( C_3 \) must have two edges in this set, while each \( C_4 \) has either two or four edges in this set. Thus we have

\[ 2p + 2q \leq s(n-s). \]

These imply that

\[ p \leq 2s(n-s) - \frac{n(n-1)}{2}. \]

But certainly \( s(n-s) \leq \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \), and since \( n \) is odd, this becomes

\[ s(n-s) \leq \frac{n^2-1}{4}. \]

Hence we obtain \( p \leq \frac{n-1}{2} \).

Next we construct appropriate designs and give their blocking sets.

First consider cases smaller than \( n = 9 \). If \( n = 3 \), the trivial design of type \( (1,0) \) can be 2-coloured. When \( n = 5 \), the set \( \{1,2\} \) is a blocking set for the design of type \( (2,1) \) with cycles

\[ \{(1,3,5), (1,2,4), (2,3,4,5)\}. \]

When \( n = 7 \), the only type \( (p,q) \) with \( p \leq 3 \) is \( (3,3) \). The set \( \{1,2,3\} \) is a blocking set for the following design of type \( (3,3) \):

\[ \{(1,4,7), (2,5,7), (3,6,7), (1,2,4,5), (1,3,4,6), (2,3,5,6)\}. \]
Now let \( n = 2m + 1 \) with \( n \geq 9 \). We shall construct a \( \{C_3, C_4\}\)-design of type \((p, q) = (m - 4k, m(m - 1)/2 + 3k)\) for each \( 4k \leq m \). Let the vertex set be

\[
\{\infty\} \cup \{(i, j) \mid 1 \leq i \leq m, j = 1, 2\};
\]

a possible blocking set will be \( \{(i, 1) \mid 1 \leq i \leq m\} \). We describe the cycles of the decomposition in the case \( k = 0 \) first. These are:

\[
B = \{(\infty, (i, 1), (i, 2)) \mid 1 \leq i \leq m\},
\]

\[
\{(i, 1), (j, 2), (i, 2), (j, 1) \mid 1 \leq i < j \leq m\}.
\]

Now the case \( k \geq 0 \) is constructed from this. For clarity we describe the case \( k = 1 \) first. Remove from \( B \) the cycles

\[
\{(\infty, (i, 1), (i, 2)) \mid i = 1, 2, 3, 4\} \cup \{(i, 1), (j, 2), (i, 2), (j, 1) \mid 1 \leq i < j \leq 4\}
\]

and replace them with a 4-cycle system of order 9, on the set

\[
\{\infty\} \cup \{(i, j) \mid 1 \leq i \leq 4, j = 1, 2\}.
\]

The result is a \( \{C_3, C_4\}\)-design of order \( n = 2m + 1 \) and type \((m - 4, m(m - 1)/2 + 3)\).

Now for arbitrary \( k \) with \( 4k \leq m \), we take the cycles in \( B \) and remove \( 4k \) cycles of length 3, say, \( \{(\infty, (i, 1), (i, 2)) \mid 1 \leq i \leq 4k\} \), and 6k cycles of length 4, namely \( \{(i, 1), (j, 2), (i, 2), (j, 1) \mid i < j\} \), for \( i, j \in \{4s - 3, 4s - 2, 4s - 1, 4s\} \), for each \( s = 1, 2, \ldots, k \). Then we replace these with the cycles from \( k \) 4-cycle systems of order 9 on the vertex sets

\[
\{\infty\} \cup \{(i, j) \mid i = 4s - 3, 4s - 2, 4s - 1, 4s, j = 1, 2\}
\]

for each \( s = 1, 2, \ldots, k \), thus removing \( 4k \) cycles of length 3 and adding \( 3k \) to the number of cycles of length 4.

The result is a 2-coloured \( \{C_3, C_4\}\)-design of type \((p, q) = (m - 4k, m(m - 1)/2 + 3k)\) for each \( 4k \leq m \).

This completes the proof of the theorem. ☐

References
