On the $\alpha$-labeling Number of Bipartite Graphs

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Abstract

We show that if $G$ is the vertex-disjoint union of graphs with $\alpha$-labelings, or if $G$ is a comet, then there exists a graph $H$ with an $\alpha$-labeling such that $H$ decomposes into two copies of $G$.

key words. $\alpha$-labeling number, bipartite, comet, hammock.

1 Introduction

Only graphs without loops and multiple edges will be considered herein. Undefined graph-theoretic terminology can be found in the textbook by Chartrand and Lesniak [1]. If $m$ and $n$ are integers we denote $\{m, m + 1, \ldots , n\}$ by $[m, n]$. Let $N$ denote the set of nonnegative integers. For any graph $G$ we call an injective function $\gamma : V(G) \rightarrow N$ a labeling of $G$. Rosa [11] called such a function $\gamma$ on a graph $G$ with $q$ edges a $\beta$-labeling if $\gamma$ is an injection from $V(G)$ into $[0, q]$ such that $\{ |\gamma(u) - \gamma(v)| : \{u, v\} \in E(G) \} = [1, q]$. A $\beta$-labeling is now more commonly called a graceful labeling. An $\alpha$-labeling is a graceful labeling having the additional property that there exists an integer $\lambda$ such that if $\{u, v\} \in E(G)$, then $\{u, v\} = \{a, b\}$, where $\gamma(a) \leq \lambda < \gamma(b)$. We will call $\lambda$ the critical value of the $\alpha$-labeling. Note that if $G$ admits an $\alpha$-labeling then $G$ is bipartite with parts $A$ and $B$, where $A = \{u \in V(G) : \gamma(u) \leq \lambda\}$, and $B = \{u \in V(G) : \gamma(u) > \lambda\}$.

Many papers have been devoted to the topic of labelings of graphs (see Gallian [5] for an excellent up-to-date survey). In particular, a conjecture of Ringel [9] and Kotzig that every tree is graceful has received much attention. In spite of many partial results, the conjecture remains open. Not every tree has an \( \alpha \)-labeling [11].

Rosa showed in [11] that \( K_{m,n} \) has an \( \alpha \)-labeling for all positive integers \( m \) and \( n \) and that \( C_m \) has an \( \alpha \)-labeling if and only if \( m \equiv 0 \pmod{4} \).

Let \( K \) and \( G \) be graphs. A \textit{G-decomposition} of \( K \) is a set \( \{G_1, G_2, \ldots, G_t\} \) of subgraphs of \( K \), each of which is isomorphic to \( G \) and such that the edge sets of the graphs \( G_i \) form a partition of the edge set of \( K \). In this case we say \( G \) \textit{divides} \( K \).

Two applications to graph decomposition make \( \alpha \)-labelings particularly attractive. Theorem 1 is proved in Rosa's original article [11]. A proof of Theorem 2 can be found in [3].

**Theorem 1** Let \( G \) be a graph with \( q \) edges, and suppose that \( G \) admits an \( \alpha \)-labeling. Then there exists a \( G \)-decomposition of the complete graph \( K_{2q^2+1} \) for every positive integer \( x \).

**Theorem 2** Let \( G \) be a graph with \( q \) edges, and suppose that \( G \) admits an \( \alpha \)-labeling. Then there exists a \( G \)-decomposition of the complete bipartite graph \( K_{2q^2+1} \) for all positive integers \( x \) and \( y \).

In [12] Snevily introduced the following terminology. We say a bipartite graph \( G \) with \( e \) edges \textit{eventually has an \( \alpha \)-labeling} provided that there exists a graph \( H \) with an \( \alpha \)-labeling that can be decomposed into \( t \) copies of \( G \). Such a graph \( H \) is called the \textit{host} graph of \( G \). The \textit{\( \alpha \)-labeling number} of \( G \) is \( G_\alpha = \min \{ t : \text{there exists a host graph } H \text{ of } G \text{ with } |E(H)| = t \cdot |E(G)| \} \).

For example it is shown in [4] that the \( n \)-cube \( Q_n \) (the Cartesian product of \( n \) copies of \( K_2 \)) can be decomposed into \( 2^{n-1} \) copies of any tree \( T \) with \( n \) edges. Since the \( n \)-cube has an \( \alpha \)-labeling for every positive integer \( n \) (see [7]), we conclude that if \( T \) is a tree with \( n \) edges, then \( T_\alpha \leq 2^{n-1} \).

Snevily showed that if \( C \) is a cycle of even length then \( C_\alpha \leq 2 \), and posed the following conjecture which was recently verified in [2].

**Conjecture 1** (Snevily [12]) \textit{If } \( G \) \textit{is a bipartite graph, then } \( G_\alpha < \infty \).

Conjecture 1 was verified by showing that every bipartite graph divides a complete bipartite graph. Recall that \( K_{m,n} \) has an \( \alpha \)-labeling for all positive integers \( m \) and \( n \).

**Theorem 3** (El-Zanati, Fu, and Shiue [2]) \textit{For each bipartite graph } \( G \) \textit{with } \( q \) \textit{edges, there exists a } \( q \)-regular bipartite graph \( H \) \textit{such that } \( H \) \textit{can be decomposed into copies of } \( G \).
Theorem 4 (Häggkvist [6]) Let $G$ be an $r$-regular bipartite graph on $2n$ vertices. Then $G$ divides $K_{r,2n,r,2n}$.

By combining the results in Theorems 3 and 4, Conjecture 1 is verified.

Corollary 5 If $G$ is a bipartite graph, then $G_\alpha$ is finite.

The bound obtained in Corollary 5 may be quite large. In this paper we show that $G_\alpha \leq 2$ for all comets and graphs which the vertex-disjoint union of graphs with $\alpha$-labelings.

Snevily expects that $T_\alpha \leq n$ if $T$ is a tree with $n$ edges. We believe that $n$ is a reasonable bound for $G_\alpha$ for any bipartite graph $G$ with $n$ edges. In particular we note the results in [6], [8], [10], [13] and [14]. We note however that we know of no example of a bipartite graph $G$ where $G_\alpha > 2$.

2 Main Results

Lemma 1 For $i \in \{1,2\}$ let $G_i$ be a graph with $q_i$ edges having an $\alpha$-labeling $\gamma_i$ with critical value $\lambda_i$. Suppose that $G_1$ and $G_2$ are vertex-disjoint. Form a graph $G$ by identifying the vertex in $G_1$ with vertex label $\lambda_1$ with the vertex in $G_2$ with vertex label 0. Then $G$ has an $\alpha$-labeling $\gamma$ satisfying $\gamma(\gamma_2^{-1}(\lambda_2)) = \lambda$, where $\lambda$ is the critical value of $\gamma$. Moreover, $\gamma(\gamma_2^{-1}(\lambda_2 + 1)) = \lambda + 1$ and $\gamma(\gamma_1^{-1}(q_1)) = q_1 + q_2$.

Proof. It can be readily checked that

$$
\gamma(v) = \begin{cases} 
\gamma_1(v) & \text{if } v \in V(G_1) \text{ and } \gamma_1(v) \leq \lambda_1 \\
\gamma_1(v) + q_2 & \text{if } v \in V(G_1) \text{ and } \gamma_1(v) > \lambda_1 \\
\gamma_2(v) + \lambda_1 & \text{if } v \in V(G_2) 
\end{cases}
$$

is an $\alpha$-labeling with the desired properties.

Similarly, a graph $G^*$ with an $\alpha$-labeling $\gamma^*$ can be formed from the graphs $G_1$ and $G_2$ in Lemma 1 by identifying $\gamma_1^{-1}(\lambda_1 + 1)$ with $\lambda_2^{-1}(q_2)$.

Lemma 2 For $i \in \{1,2\}$ let $G_i$ be a graph with $q_i$ edges having an $\alpha$-labeling $\gamma_i$ with critical value $\lambda_i$. Suppose that $G_1$ and $G_2$ are vertex-disjoint. Form a graph $G^*$ by identifying the vertex in $G_1$ with vertex label $\lambda_1 + 1$ with the vertex in $G_2$ with vertex label $q_2$. Then $G^*$ has an $\alpha$-labeling $\gamma^*$ with critical value $\lambda^* = \gamma^*(\gamma_2^{-1}(\lambda_2))$. Moreover, $\gamma^*(\gamma_2^{-1}(\lambda_2 + 1)) = \lambda^* + 1$ and $\gamma^*(\gamma_1^{-1}(q_1)) = q_1 + q_2$. 

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Proof. It can be readily checked that

\[
\gamma^*(v) = \begin{cases} 
\gamma_1(v) & \text{if } v \in V(G_1) \text{ and } \gamma_1(v) \leq \lambda_1 \\
\gamma_1(v) + q_2 & \text{if } v \in V(G_1) \text{ and } \gamma_1(v) > \lambda_1 \\
\gamma_2(v) + \lambda_1 + 1 & \text{if } v \in V(G_2)
\end{cases}
\]

is an \(\alpha\)-labeling with the desired properties.

The following theorem is an immediate consequence of the above two lemmas.

**Theorem 6** For \(i \in [1, n]\) let \(G_i\) be a graph with \(\alpha\)-labeling \(\gamma_i\) with critical value \(\lambda_i\). Suppose the graphs \(G_i\) are pairwise vertex-disjoint. Form a graph \(G\) by either identifying \(\gamma_i^{-1}(\lambda_i)\) with \(\gamma_{i+1}^{-1}(0)\) or identifying \(\gamma_i^{-1}(1 + \lambda_i)\) with \(\gamma_{i+1}^{-1}(q_{i+1})\), \(1 \leq i \leq n - 1\). Then \(G\) has an \(\alpha\)-labeling.

**Theorem 7** For \(i \in [1, n]\) let \(G_i\) be a graph with an \(\alpha\)-labeling. Suppose the graphs \(G_i\) are mutually vertex-disjoint. Let \(G = G_1 \cup G_2 \cup \ldots \cup G_n\). Then \(G_\alpha \leq 2\).

Proof. We will present a proof for \(n = 2\). The idea generalizes for \(n > 2\). For \(i \in \{1, 2\}\) let \(G_i\) be a graph with \(q_i\) edges with the \(\alpha\)-labeling \(\gamma_i\) with critical value \(\lambda_i\), and let \(G_i^*\) be an isomorphic copy of \(G_i\) with the (corresponding) \(\alpha\)-labeling \(\gamma_i^*\). Form a graph \(G'\) by linking the graphs in the sequence \(G_1, G_2, G_1^*, G_2^*\) by identifying the ordered pairs of vertices \((\gamma_1^{-1}(\lambda_1), \gamma_2^{-1}(0)), (\gamma_2^{-1}(\lambda_2 + 1), \gamma_2^*^{-1}(q_2))\), and \((\gamma_2^{-1}(\lambda_2), \gamma_1^*^{-1}(0))\). Since \(G\) is isomorphic to both \(G_1 \cup G_2\) and \(G_1^* \cup G_2^*\), we have \(G\) divides \(G'\) and thus \(G_\alpha \leq 2\).

Note that a graph with \(q\) edges and more than \(q + 1\) vertices cannot have even a \(\beta\)-labeling.

**Corollary 8** If every component of a forest \(G\) with more than one component has an \(\alpha\)-labeling, then \(G_\alpha = 2\).

3 Hammocks

Let \(H_{k,m}\) be the graph with vertices \(u, v, x_{ij}, 1 \leq i < m, 1 \leq j \leq k\), and for \(1 \leq j \leq k\) the edges \(\{v, x_{i1}\}, \{x_{ij}, x_{i+1,j}\}, 1 \leq i < m - 1, \text{ and } \{x_{m-1,j}, u\}\). We call such a graph a hammock. Note that \(H_{k,m}\) consists of \(k\) paths with \(m\) edges, joined at their endpoints. (See Figure 1.)

**Theorem 9** Let \(k\) and \(n\) be positive integers, where if \(n > 2\), then \(k\) is odd. Then the hammock \(G = H_{k,2n}\) has an \(\alpha\)-labeling such that the label on \(u\) is 0 and the label on \(v\) is the critical value.
Proof. We define a labeling $\gamma$ of the vertices of $G$ by $\gamma(u) = 0$, $\gamma(v) = kn$, and, for $1 \leq i < 2n, 1 \leq j \leq k$,

$$\gamma(x_{ij}) = \begin{cases} 
    k(2n-(i-1)/2) - j + 1 & \text{if } i \text{ is odd} \\
    k(1+i/2) - 2j + 1 & \text{if } i \text{ is even}.
\end{cases}$$

We will show that $\gamma$ is an $\alpha$-labeling with $\lambda = kn$. First we show that $\gamma$ is one-to-one. Notice that $\gamma$ is one-to-one on the vertices $x_{ij}$ with $i$ odd, since increasing $i$ by 2 decreases $\gamma(x_{ij})$ by $k$.

If $n \leq 2$ there is at most one even $i$ with $1 \leq i < 2n$ and so $\gamma$ is one-to-one on the $x_{ij}$ with $i$ even. Otherwise we can assume $k$ is odd. But if $k(1+i/2) - 2j + 1 = k(1+i/2) - 2j + 1$ with $i$ and $I$ even and $j$ and $J$ in $[1,k]$, then $k$ divides $2(j-J)$, and so $j = J$ and $i = I$. Thus $\gamma$ is one-to-one on the $x_{ij}$ with $i$ even also.

Now if $i$ is odd then $\gamma(x_{ij}) \in [k(2n-(2n-1)/2) - k+1, k(2n-1)] = [kn+1, 2kn]$, while if $i$ is even then $\gamma(x_{ij}) \in [k(1+1) - k+1, k(1+(2n-2)/2) - 2 + 1] = [1, kn-1]$. Thus the values of $\gamma(x_{ij})$ with $i$ odd do not overlap those with $i$ even or with $\gamma(u) = 0$ or $\gamma(v) = kn$.

Now we will show that the edge labels are exactly $[1, 2kn]$. First consider the edges $\{x_{ij}, x_{i-1,j}\}$ with $i$ odd, $3 \leq i \leq 2n-1$. Then

$$\gamma(x_{ij}) - \gamma(x_{i-1,j}) = k(2n-(i-1)/2) - j + 1 - (k(1+(i-1)/2) - 2j + 1) = k(2n-i) + j,$$

giving the set of labels $k\{1, 3, 5, \ldots, 2n-3\} + j$. We also have the edges $\{x_{ij}, x_{i+1,j}\}$ with $i$ odd, $1 \leq i \leq 2n-3$. Then

$$\gamma(x_{ij}) - \gamma(x_{i+1,j}) = k(2n-(i-1)/2) - j + 1 - (k(1+(i+1)/2) - 2j + 1) = k(2n-i-1) + j,$$

giving the labels $k\{2, 4, \ldots, 2n-2\} + j$. Taking these two sets together gives $[k+1, k(2n-1)]$. Finally, the edges $\{u, x_{2n-1,j}\}$ have labels $\gamma(x_{2n-1,j}) - \gamma(u) = k(2n-(2n-2)/2) - j + 1 - kn = k + 1 - j$, giving the set $[1, k]$, while the edges $\{u, x_{1j}\}$ have labels $\gamma(x_{1j}) - \gamma(u) = k(2n) - j + 1 - 0$, giving the set $[k(2n-1) + 1, 2kn]$. Note that the vertices $u, v$ and $x_{ij}$ with $i$ even have labels not exceeding $kn$, while the other vertices have labels greater than $kn$. Thus $\gamma$ is an $\alpha$-labeling with critical value $\lambda = kn$.

Not all hammocks have $\alpha$-, or even $\beta$-labelings. Indeed Rosa [11] proved that a graph in which the degree of every vertex is even has no $\beta$-labeling if its number of edges is congruent to 1 or 2 modulo 4. Thus $H_{k,m}$ has no $\beta$-labeling if $k \equiv 2 \pmod{4}$ and $m$ is odd.
4 Comets

Let $S_{k,n}$ be the graph with vertices $u$ and $x_{i,j}$, $1 \leq i \leq n$, $1 \leq j \leq k$, and for $1 \leq j \leq k$ the edges $\{u, x_{1,j}\}$ and $\{x_{i,j}, x_{i+1,j}\}$, $1 \leq i < n$. We call such a graph a comet. The comet $S_{k,n}$ consists of $k$ paths with $n$ edges, joined at an endpoint of each path. Not every comet has an $\alpha$-labeling. For example, $S_{k,2}$ has no $\alpha$-labeling if $k > 2$ (see [11]). Nevertheless we can prove the following.

**Theorem 10** The $\alpha$-labeling number of any comet is $\leq 2$.

**Proof.** Notice that the hammock $H_{k,2n}$ can be decomposed into 2 copies of the comet $S_{k,n}$. Thus if $k$ is odd this result follows immediately from Theorem 9.

If $k$ is even we will get the same result by adding two paths of length $n$ to the graph $H_{k-1,2n}$. It is well-known that the path $P$ with $n$ edges and vertices $w_0, w_1, \ldots, w_n$ has an $\alpha$-labeling $\gamma$ given by

$$\gamma(w_i) = \begin{cases} \frac{i}{2} & i \text{ even} \\ \frac{n-i+1}{2} & i \text{ odd.} \end{cases}$$

Let the path $P^*$ with vertices $w_0^*, w_1^*, \ldots, w_n^*$ be an isomorphic copy of $P$ and define an $\alpha$-labeling $\gamma^*$ on $P^*$ by

$$\gamma^*(w_i^*) = \begin{cases} \gamma(w_i) & n \text{ even} \\ n - \gamma(w_i) & n \text{ odd.} \end{cases}$$
Figure 2: An $\alpha$-labeling of a graph that decomposes into two copies of $S_{6,3}$

We note that $\gamma(w_0) = 0$, while $\gamma^*(w^*_{n-1}) - \gamma^*(w^*_n) = 1$, so $\gamma^*(w^*_n) = \lambda^*$, the critical value of $\gamma^*$.

By Theorem 9 the hammock $H_{k-1,2n}$ has an $\alpha$-labeling for which the label on $u$ is 0, while the label on $v$ is the critical value. Now we construct a new graph by attaching paths of length $n$ at $u$ and $v$. Specifically, by Theorem 6 we get a graph with an $\alpha$-labeling by identifying $w^*_n$ with $u$, and $v$ with $w_0$. (See Figure 2.) Clearly this graph can be decomposed into two copies of the comet $S_{k,n}$.

References


