A STUDY OF TYPENUMBER IN BOOK-EMBEDDING

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Abstract

The type of a vertex \( v \) in a \( p \)-page book-embedding is the \( p \times 2 \) matrix of nonnegative integers

\[
\tau(v) = \begin{pmatrix}
  l_{v,1} & r_{v,1} \\
  \vdots & \vdots \\
  l_{v,p} & r_{v,p}
\end{pmatrix},
\]

where \( l_{v,i} \) (respectively, \( r_{v,i} \)) is the number of edges incident to \( v \) that connect on page \( i \) to vertices lying to the left (respectively, to the right) of \( v \). The typenumber of a graph \( G \), \( T(G) \), is the minimum number of different types among all the book-embeddings of \( G \). In this paper, we disprove the conjecture by J. Buss et. al. which says for \( n \geq 4 \), \( T(L_n) \) is not less than 5 and prove that \( T(L_n) = 4 \) for \( n \geq 3 \).

1 Introduction and Basic Properties

A book is a set of half-planes (the pages of the book) that share a common boundary line (the spine of the book). An embedding of a simple undirected graph of \( G \) (a pair of vertices are connected by at most one edge) in a book consists of an ordering of the vertices of \( G \) along the spine (horizontal line) of the book, together with an assignment of each edge of \( G \) to a page of the book, in which edges assigned to the same page do not cross.

There are three germane measure of the quality of a book-embedding: the thickness (number of pages) of the book, the individual and cumulative

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widths of the pages, and the number of distinct vertex types. Throughout
of this paper we shall focus on the study of the third measure.

Given a \( p \)-page book-embedding of a graph \( G \), each vertex \( v \) of \( G \) has
an associated \( p \times 2 \) matrix of nonnegative integers, called its type,

\[
\tau(v) = \begin{pmatrix}
    l_{v,1} & r_{v,1} \\
    \vdots & \vdots \\
    l_{v,p} & r_{v,p}
\end{pmatrix},
\]

where \( l_{v,i} \) (respectively, \( r_{v,i} \)) is the number of edges incident to \( v \) that con-
nect on page \( i \) to vertices lying to the left (respectively, to the right) of
\( v \). Thus for each graph of order \( n \) and each \( p \)-page book-embedding, there
are \( n \) types, one for each vertex and two types are different provided that
the two matrices are not equal. It's interesting to know : among all the
book-embeddings of \( G \) what is the minimum number of different vertex
types, \( T(G) \). For its application, the number of types in a book-embedding
relates to the amount of logic necessary to realize fault-tolerant arrays of
processors using one specific design methodology. The methodology views
the desired array as an undirected graph, with vertices representing pro-
cessing elements and edges representing communication links; the design
process operates in two stages: first, the graph representing the desired
array is embedded in a book; then, the book-embedding is converted to an
efficient fault-tolerant layout of the array. The significance of the notation
of vertex type is that the type of a vertex "tell" it what role to play in the
fault-free processor array. Thus, the base-two logarithm of the number of
vertex types is the number of control bits per processing element needed
to configure the array to its fault-free format. \cite{2} \cite{3}

In any book-embedding of \( G \), there are two specific vertex types in the
embedding : source and sink. A nonzero vertex type for a vertex \( v \) is a
source if all \( l_{v,i} = 0 \) and is a sink if all \( r_{v,i} = 0 \). Clearly, every book-
embedding of a graph has at least one source and one sink.

Let \( N_i = \{ v \in V(G) \mid \deg_G(v) = i \} \) and \( T_i \) be the different types count-
ing all the vertices in \( N_i \), i.e., \( T_i = |\{\tau(v) \mid v \in N_i\}| \). By considering
the relation between the degree of \( v \) and the \( p \times 2 \) matrix, \( \tau(v) \), we have the
followings.

**Lemma 1.1.** \cite{1} For each \( v \in V(G) \), \( \deg_G(v) = \sum_{i=1}^{p} (l_{v,i} + r_{v,i}) \), hence
\( \tau(u) \neq \tau(v) \) provided that \( \deg_G(u) \neq \deg_G(v) \). Thus \( T(G) = \sum_{i \in D_\circ} T_i \).

**Lemma 1.2.** \( \sum_{v \in V(G)} \sum_{i=1}^{p} l_{v,i} = \sum_{v \in V(G)} \sum_{i=1}^{p} r_{v,i} \).
Lemma 1.3. [1] A connected graph $G$ with at least two vertices has type-number two if and only if $G$ is a star.

A lattice graph $L_{m,n}$ is a graph with vertex set $\{(a,b) | a \in \mathbb{Z}_m \text{ and } b \in \mathbb{Z}_n\}$ and edge set $\{(a,b), (c,d) \mid a = c \text{ and } |b - d| = 1 \text{ or } b = d \text{ and } |a - c| = 1\}$. While $m = 2$ or $n = 2$, it's called a ladder, denoted by $L_2(L_2)$.

Proposition 1.4. [1] Restricting to 1-page book-embedding of $L_n$ we have: $T(L_2) = 2$, $T(L_3) = 3$, $T(L_4) = 4$, and $T(L_n) = 5$, for $n \geq 4$.

In [1], the authors conjectured that additional pages for $L_n$, $n \geq 4$, will not lower its typenumber below 5. Unfortunately, this conjecture is not true as we shall see in section 2.

2 Main Result

A book-embedding graph $G_K$ corresponding to a book-embedding $K$ of $G$ is a colored (not necessarily proper) digraph such that (i) an arc $uv \in A(G_K)$ if and only if $uv \in E(G)$ and $u$ is to the left of $v$ along the spine in the embedding where the two sides are fixed, and (ii) two arcs have the same color if and only if the corresponding edges are embedded by $K$ on the same page.

For convenience, we use $G_K(\Pi)$ to denote the subgraph of $G_K$ which is induced (edge) by the set of edges whose colors are in $\Pi$. If $\Pi = \{c_1, c_2, \cdots, c_k\}$, we also use $G_K(c_1, c_2, \cdots, c_k)$ to denote the subgraph. Now, we have the following results.

Lemma 2.1. Let $G_K$ be a book-embedding graph of $G$ corresponding to a book-embedding $K$. Then, for each color $c$, each cycle in $G_K(c)$ is almost a directed cycle except exactly one arc.

Proof. First, we observe that a directed path $(a_0, a_1, \cdots, a_t)$ in $G_K(c)$ gives the relation that $a_i$ is to the left of $a_j$ if and only if $0 \leq i < j \leq t$. Furthermore, any vertex $v$ lies between $a_i$ and $a_{i+1}$ can be incident to neither $a_0$ nor $a_t$ for any $i \geq 0$ and $i < j$ because they are at the same page in the book-embedding. Now, given a cycle $C$ in $G_K(c)$ and choose $P = (a_0, a_1, \cdots, a_t)$ to be the longest directed path in $C$. If

(1). $t = 1$.

So $P = a_0, a_2, a_3, \cdots, a_t$. Hence, the type number of this cycle $C$ is 2, a contradiction to Lemma 1.3.

(2). $t \geq 2$ and the length of $C$ is $t + 1$.

Obviously, $C$ is almost a directed cycle except exactly one arc.
(3). $t \geq 2$ and the length of $C$ is large than $t + 1$

Let $v(\neq a_1)$ be incident to $a_0$ in $C$ and $v$ must be to the right of $a_0$ because $P$ is the longest directed path. By the observation above, $v$ must be to the right of $a_t$ too. Similarly, let $v'(\neq a_{t-1})$ be incident to $a_t$ in the cycle, we can obtain the relation that $v'$ is to the left of $a_1$. So $v \neq v'$ and a cross happens between $a_0 v$ and $v' a_t$, a contradiction.

Then the only possibility is that the cycle $C$ is almost a directed cycle except exactly one arc.

Now, we consider the graph induced by two colors.

**Lemma 2.2.** Let $G_K$ be a book-embedding graph of $G$ corresponding to a book-embedding $K$. Then there doesn't exist a cycle $C$ in $G_K(c_1, c_2)$ such that the arcs of the same color have the same orientation and the number of vertices which are incident with 2 colors edges are at least 4.

**Proof.** Suppose not. Let $C$ be such a cycle in $G_K(c_1, c_2)$ and $C$ can be partitioned into

$$P_1(v_1, v_2)P_2(v_2, v_3)P_3(v_3, v_4) \cdots P_{k-1}(v_{k-1}, v_k)P_k(v_k, v_1)$$

such that $P_i$ and $P_{i+1}$ are two directed paths with the opposite orientations and colors. And obviously $k$ is an even number and $k \geq 4$. By contracting each $P_i$ to an arc $\overline{v_i v_i' + 1}$ (or $\overline{b_i, v_{i+1}}$), the cycle $C$ becomes to a new cycle $C'$

$$\overline{v_1 v_2} \overline{b_2 v_3} \overline{v_3 v_4} \cdots \overline{v_{k-1} v_k} \overline{b_k v_1}.$$ 

Since $C$ is a subgraph of the book-embedding graph $G_K$, we can obtain a 2 page book-embedding of a cycle which is the underlying graph of $C$. And it's easy to see that the underlying graph of the contracted cycle $C'$ can be embedded in this 2 page book and each vertex is either a source $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ or a sink $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. So the typenumber of this cycle is two. But it's not a star, a contradiction.

**Theorem 2.3.** $T(L_3) = 4$

**Proof.** By Figure 2, $T(L_3) \leq 4$. We suppose $T(L_3) < 4$. Since $L_3$ isn't a star, by lemma 1.3, $T(L_3) \geq 3$. Assume that $T(L_3) = 3$ and we have two cases to consider now.
(1). \( T_2 = 2, T_3 = 1 \):

Since \( Aa, Cc \in E(L_3) \), one of \( \{A, a\} \) is a source and the other is a sink. So is \( C, c \). Since \( Bb \in E(L_3) \), W.L.O.G., we may let \( \sum l_{B,i} = \sum l_{b,i} = 2 \). Hence \( \sum \sum l_{v,i} \neq \sum \sum r_{v,i} \) and we have a contradiction.

(2). \( T_2 = 1, T_3 = 2 \):

The only vertex type of vertices of \( N_2 \) is \( \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \). Then W.L.O.G. Figure 3 show the possible book-embedding graph only can be the following two types:

(i) ![Diagram 1](image)

(ii) ![Diagram 2](image)

Figure 3: The six arcs must have the same color.
For each type, there exists a one-color cycle which is forbidden in Lemma 2.1, a contradiction. So we conclude that $T(L_3) = 4$.

\[ \text{Theorem 2.4. } T(L_n) = 4 \text{, } n \geq 4. \]

\[ \text{Proof. By Figure 4, } T(L_n) \leq 4, \text{ since the vertex types are } \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \]

![Figure 4: A book-embedding of $T(L_n)$](image)

Assume that $T(L_n) < 4$. There are Three cases to consider:

(1). $T_2 = 1, T_3 = 1$ (i.e. $T(L_n) = 2$)
   It's trivially impossible because $L_n$ isn't a star.

(2). $T_2 = 2, T_3 = 1$ (i.e. $T(L_n) = 3$)
   By the same reason as (1) in Theorem 2.3. $\sum \sum l_{v,i} \neq \sum \sum r_{v,i}$, a contradiction.

(3). $T_2 = 1, T_3 = 2$ (i.e. $T(L_n) = 3$)
   Obviously, the only vertextype of vertices of $N_2$ is $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, and the only two vertex types of vertices of $N_3$ are a source and a sink.
Figure 5: Each subgraph contains a two-color cycle forbidden in Lemma 2.2.

W.L.O.G., the possible book-embedding graph must contain one of the following three induced subgraphs:

Since, there exists a two-color cycle for each subgraph which is forbidden in Lemma 2.2, a contradiction. So $T(L_n) = 4$.

References


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